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SYMPLECTIC TOPOLOGY AS THE GEOMETRY OF ACTION FUNCTIONAL. I
—RELATIVE FLOER THEORY ON THE COTANGENT BUNDLE

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1. Introduction and the main result

In the late 70's or the beginning of the 80's, Eliashberg proved the following theorem, which first indicated the existence of *symplectic topology* that is supposed to be finer than *differential topology.*

C^0-rigidity theorem [Eliashberg]. *The group $Symp_\omega(P)$ of symplectic diffeomorphisms on a symplectic manifold (P, w) is C^0-closed in Diff (P).*

Eliashberg's original proof [12] relies on a structure theorem on the *combinatorial* structure of the *wave front set* of certain Legendrian submanifolds in the one-jet bundle. The complete detail of the proof of this structure theorem, however, has not been published in the literature. The heart of his proof is some kind of non-squeezing theorem, which he proved using the above structure theorem. In a seminal paper [28] in 1985, Gromov introduced the *elliptic techniques of pseudo-holomorphic curves* and proved, among many other things, the following non-squeezing theorem.

Non-squeezing theorem [Gromov]. *Let $B^{2n}(R) \subset \mathbb{C}^n$ be the standard R-ball in $\mathbb{C}^n$ and w_0 be the canonical symplectic structure on $\mathbb{C}^n$. Then there is a symplectic embedding*

$$\phi : (B^{2n}(R), w_0) \to (Z^{2n}(r), w_0)$$

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iff $R \leq r$. *Here* $Z^{2n}(r) = B^2(r) \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$.

This non-squeezing theorem is the beginning of the so called *symplectic capacity theory* and in fact, the existence of any symplectic capacity function on the set of open sets in $\mathbb{C}^n$ will provide a relatively straightforward proof of the C^0-rigidity theorem using Eliashberg's argument in Section 2.3 [12]. Shortly after Gromov's work [28], Ekeland and Hofer [15], [16] used the *variational method of the existence theory of* periodic orbits of Hamiltonian systems on $\mathbb{C}^n$ to construct other symplectic capacities. (We refer to [31] for a detailed exposition on the symplectic capacity theory.) This variational theory was culminated by Hofer [30] into the so called *Hofer's geometry* on the group $\mathcal{D}^c_\omega(P)$ of Hamiltonian diffeomorphisms. The above C^0-rigidity is encoded into a remarkable bi-invariant (Finsler) distance on the group. In [32], Lalonde and McDuff set the stage of Hofer's geometry on any symplectic manifold (P, w) by proving that Hofer's pseudo-norm on $\mathcal{D}^c_w(P)$ is nondegenerate on any (P, w). Unlikely from Hofer's proof on $\mathbb{C}^n$, Lalonde and McDuff used the Gromov theory of pseudo-holomorphic curves together with an ingenious method of constructing optimal symplectic embedding of balls. They also made more detailed investigation of Hofer's geometry on (P, ω) in [33] e.g, concerning the geodesics on the geometry.

On the other side of symplectic geometry, Arnold [1] in the 1960's, first predicted the existence of *Lagrangian intersection theory* (on the cotangent bundle) as the intersection theoretic version of the Morse theory and posed the celebrated *Arnold's conjecture.* We would like to recall that the intersection theoretic version of the degree theory of generic vector fields is *the Lefschetz intersection theory.* (We refer to our survey paper [42] for more explanations on this aspect.)

Arnold's conjecture (on T^*M). *Let M be a compact n-manifold and*

$$L_0 = \phi(o_M), \qquad L_1 = o_M,$$

where $o_M \subset T^*M$ is the zero section, and ϕ is a Hamiltonian diffeomorphism. Then

$$\begin{aligned} \#(L_0 \cap L_1) &\geq CRN(M) \qquad \textit{for the transverse case} \\ &\geq CR(M) \qquad \textit{in general,} \end{aligned}$$

where

$$CR(M) := \inf_f \#\{\mathrm{Crit}(f) \mid f \in C^\infty(M)\},$$
$$CRN(M) := \inf_f \#\{\mathrm{Crit}(f) \mid f \in C^\infty(M) \textit{ is Morse}\}.$$

Because of the lack of understanding of the invariants $CRN(M)$ or $CR(M)$, this conjecture is widely open. However, its cohomological version was proven by Hofer [29] using the *direct approach* of the classical variational theory of *the action functional* which was inspired by Conley-Zehnder's earlier proof [11] of the (cohomological version) of Arnold's conjecture on the number of fixed points of Hamiltonian diffeomorphisms. Although the basic idea in [29] is simple in the presence of [11], carrying out all the details of the direct approach involves many tedious computations partly due to the lack of global coordinates on T^*M (except the case where $M = T^n$ for which Chaperon [8] had earlier proved Hofer's result on T^n using the idea of *broken geodesics*). Shortly after, a much simpler proof using a *finite dimensional reduction of the action functional* with the idea of broken geodesics was given by Laudenbach and Sikorav [34]. This approach has been further developed by Sikorav [53] and then culminated into the Viterbo's theory of *generating functions* [56]. This finite dimensional approach completely eliminates the infinite dimensional analysis (both the elliptic theory and the variational theory) but instead uses a rather sophisticated topological machinery and geometric constructions. However, this approach still captures the C^0-rigidity theory and most of the proofs involved are rather straightforward as Viterbo himself put it in [56]. The way how Viterbo used generating functions in the applications to symplectic topology is through the construction of certain symplectic invariants of Lagrangian submanifolds by the (finite dimensional) critical point theory of generating functions, which up to normalization, depends only on the Lagrangian submanifold that is generated by the generating function used, but not on the individual generating function. The relation of these invariants to the (Hofer's) geometry (on the space) of Lagrangian submanifolds becomes obscure (or at least not apparent) during this process.

One of the main goals of the present paper is to introduce the Floer theory of Lagrangian intersections as the major tool in the symplectic topology and to attempt to incorporate different approaches to symplectic topology mentioned above in one framework (Eliashberg's wave front theory, Hofer's geometry and Viterbo's techniques of generating functions) and to lay the foundation not only to serious applications of the Floer theory and but also to the future development in the symplectic topology. Recall that Floer introduced in [18] the *Floer homology* to study the Lagrangian intersection theory, more precisely the Arnold conjecture of Lagrangian submanifolds $L \subset P$ with $\pi_2(P, L) = \{e\}$. For example, Hofer's theorem mentioned above is a special case of Floer's [18], [20] (at least up to the orientation problem which we now solve in the present paper), if we set $L_0 = \phi(o_M)$, $L_1 = o_M$ in the cotangent bundle. One crucial new point in our Floer theory in the present paper is a careful study of the filtration present in the Floer homology. This point was previously used by Floer and Hofer [23] in their symplectic homology theory but it is the first time to be carefully studied for the case of Lagrangian submanifolds. We will show that this relative case involves many new interesting geometric and algebraic constructions.

Floer [18] defined the *Floer homology* by considering the Cauchy-Riemann equation for the maps $u : \mathbb{R} \times [0, 1] \to T^*M$ (we will concern only the cotangent bundle in this paper),

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\frac{\partial u}{\partial t} = 0, \\ u(\tau, 0) \in L_0 = \phi(o_M), \\ u(\tau, 1) \in L_1 = o_M, \end{cases} \tag{1.1}$$

which becomes the equation of a L^2-type gradient flow of some real valued functional (a variation of the classical action functional), *when it exists*, on the space of paths

$$\Omega(L_0, L_1) = \{\gamma : [0, 1] \to T^*M \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}.$$

(See Section 2.3 for its precise definition in the cases we study in this paper.) We call this functional *Floer's (action) functional* and denote it by $\underline{a}$. This is defined up to addition of constant. The critical points of $\underline{a}$ correspond to the intersections of L_0 and L_1. We call this version of the Floer theory for the Lagrangian intersection the *geometric version*.

On the other hand, *when the Hamiltonian H generating ϕ (or L_0) is given* (i.e., $L_0 = \phi_H^1(o_M), L_1 = o_M$), the intersections of $L_0 = \phi_H^1(o_M)$ and $L_1 = o_M$ have one to one correspondence with the solutions of Hamilton's equation

$$\begin{cases} \dot{z} = X_H(z), \\ z(0) \in o_M, \quad z(1) \in o_M, \end{cases} \tag{1.2}$$

which are nothing but the critical points of the classical action functional

$$\mathcal{A}_H(\gamma) = \int \gamma^*\theta - \int_0^1 H(\gamma(t), t)dt$$

on the space of paths

$$\Omega(M) = \{\gamma : [0,1] \to T^*M \mid \gamma(0),\ \gamma(1) \in o_M\}.$$

Here we denote by θ the canonical one-form on T^*M. Just like for the Floer's functional, one can study the intersection problem of $\phi_H^1(o_M)$ and o_M by considering the L^2-type gradient flow of $\mathcal{A}_H$ on $\Omega(M)$ (see Section 2.4 for the precise set-up), whose equation becomes

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0, \\ u(\tau, 0) \in o_M,\ u(\tau, 1) \in o_M. \end{cases} \tag{1.3}$$

We call this version of the Floer theory the *dynamical version*.

One can easily transform (1.3) into the form of (1.1) through the map

$$\begin{aligned} &u \mapsto \phi_H^1(\phi_H^t)^{-1}u := \widetilde{u}, \\ &J \mapsto (\phi_H^t(\phi_H^1)^{-1})^*J = J^H. \end{aligned} \tag{1.4}$$

The advantage of (1.1) against (1.3) is that (1.1) involves only L_0 and L_1 not the diffeomorphism ϕ_H^1, and hence the geometry of the solutions will depend only on L_0 and L_1, although this time the almost complex structure apparently depends on H. Here we use the important fact that *the space of compatible almost complex structures is symplectically invariant and contractible*. The importance of this fact has been used often in symplectic geometry starting from [28], but not as extensively as in this paper. One important philosophy of ours is *to transform the difficult problem of isotoping Lagrangian submanifolds into the trivial problem of isotoping compatible almost complex structures*, which enables us to easily get around, in our approach, the nontrivial question of the uniqueness problem in Viterbo's approach of generating functions. On the other hand, $\mathcal{A}_H$ has a natural connection to both Hofer's geometry and to generating functions. This is because the classical action functional $\mathcal{A}_H$ on the space of paths *free at the final time*

$$\Omega = \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in o_M\}$$

is a *canonical generating function* in terms of the fibration

$$\Omega \to M; \;\; p(\gamma) := \pi(\gamma(1)).$$

One crucial observation of ours is that although the Floer theory of $\mathcal{A}_H$ on the whole space Ω cannot be done due to some analytical obstruction, it can be done nicely on the subset

$$\Omega(S) = \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in o_M, \; \gamma(1) \in N^*S\} \subset \Omega$$

for any (compact) submanifold $S \subset M$. The critical points of $\mathcal{A}_H|_{\Omega(S)}$ is the set of solutions of

$$\begin{cases} \dot z = X_H(z) \\ z(0) \in o_M, \quad z(1) \in N^*S. \end{cases} \tag{1.5}$$

The gradient flow of the restricted functional $\mathcal{A}_H\big|_{\Omega(S)}$ with respect to a suitable metric on $\Omega(S)$ becomes

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S = \text{ the conormal bundle of } S, \end{cases} \tag{1.6}$$

which is an *elliptic boundary value problem* of the (perturbed) Cauchy-Riemann equation. (1.3) is a special case of (1.6) where $S = M$. Following Floer's standard construction, we now form the set

$$CF(H, S) = \text{ the set of solutions of (1.5)}$$

and study the moduli space $\mathcal{M}_J(H, S)$ of solutions of (1.6). There are two advantages of our dynamical version of the *relative Floer theory* (i.e., Floer theory for Lagrangian intersections) against the more standard geometric version as in [18] or [38]. *First*, there exists a canonical $\mathbb{Z}$-grading on $CF(H, S)$ that is provided by the Maslov index canonically assigned to the solutions of (1.5) (see Theorem 5.1 for the precise statement). *Secondly*, $\mathcal{M}_J(H, S)$ carries coherent orientations (see Theorem 5.2), which enables us to define the relative Floer homology *with arbitrary coefficients*. We would like to emphasize that for the standard geometric version as in [18], there is no canonical grading (see [19] the definition of a non-canonical grading). Furthermore, in the general relative Floer theory as in [18] or [38] unlikely from the *non-relative* Floer theory (i.e., the Floer theory for the Hamiltonian diffeomorphisms) as in [21] and [22], each Floer cell itself $\mathcal{M}_J(z^\alpha, z^\beta)$ is not necessarily orientable, let alone talking about the existence of coherent orientations. This has forced us to look at the relative Floer theory only with $\mathbb{Z}_2$-coefficients so far. Our solutions of both the canonical grading and the orientability of the Floer cells $\mathcal{M}_J(z^\alpha, z^\beta)$ soly depend on the special circumstance in the framework of the dynamical version where we are looking at the fixed Lagrangian submanifolds, *the co-normal bundles* in the *cotangent bundle with* a given Hamiltonian generating the Lagrangian submanifold $L = \phi_H(o_M)$.

Theorem I. *Let us choose any generic choice of (H, S, J) inside its isotopy class $[H, S, J] \cong [S]$.*

(1) *There exists a canonical $\mathbb{Z}$-grading on $CF(H, S : M)$. Denote by $CF_*(H, S : M)$ the $\mathbb{Z}$-graded module (with arbitrary coefficients) generated by $CF(H, S : M)$.*

(2) *$\mathcal{M}_J(H, S)$ carries a coherent orientation, denoted by σ that is compatible to the gluing procedure in the sense of [22], [23], and so there exists a boundary map*

$$\partial_{(H,J)} = \partial^\sigma_{(H,J)} : CF_*(H, S : M) \to CF_*(H, S : M),$$

that has degree -1 and hence, we can define the Floer homology

$$HF_*(H, S, J : M) = HF^\sigma_*(H, S, J : M)$$

with arbitrary coefficients for such (H, S, J). We denote the set of coherent orientations by $\mathrm{Or}([H, S, J]) =: \mathrm{Or}([S] : M)$, whose precise definition we will refer to Section 5 below and to [22]. As the notation suggests, this set depends only on the isotopy class $[H, S, J] \cong [S]$.

(3) *For each fixed coherent orientation σ and for each generic pair $(H^\alpha, S^\alpha, J^\alpha)$, $(H^\beta, S^\beta, J^\beta)$ isotopic to each other, there exists a canonical isomorphism*

$$h^\sigma_{\alpha\beta} : HF^\sigma_*(H^\alpha, S^\alpha, J^\alpha : M) \to HF^\sigma_*(H^\beta, S^\beta, J^\beta : M),$$

that preserves the canonical grading.

(4) *Furthermore, there exists a coherent orientation σ, which we call the canonical coherent orientation and with respect to which there exists a canonical isomorphism*

$$F_{(H,S,J)} : H_*(S, \mathbb{Z}) \to HF^\sigma_*(H, S, J : M),$$

which in particular shows that $HF_*(H, S, J : M)$ *is non-trivial.*

We will always suppress the canonical coherent orientation σ provided in (4) from notation throughout the paper except when we need to emphasize the dependence on the coherent orientation.

Using the fact that (1.6) is the gradient flow of $\mathcal{A}_H\big|_{\Omega(S)}$ with respect to a L^2-type metric on $\Omega(S)$ (depending on J), we can also study the filtration, and define the relative homology groups

$$HF_*^{[a,b)}(H, S, J : M) \qquad \text{for } b > a.$$

Furthermore, we show (Theorem 5.4) that for fixed (H, S), there exists a canonical isomorphism

$$HF_*^{[a,b)}(H, S, J^\alpha : M) \to HF_*^{[a,b)}(H, S, J^\beta : M)$$

for two generic J_α and J_β. Using these, one can define a number

$$\rho(H, S, J) = \inf_\lambda \{\lambda \mid HF_*^{(-\infty,\lambda)}(H, S, J, M) \\ \to HF_*(H, S, J : M) \text{ is surjective}\},$$

and prove that $\rho(H, S, J)$ is independent of J. We denote the common number by $\rho(H, S)$.

Theorem II. (1) *$\rho(H, S)$ is a (finite) critical value of $\mathcal{A}_H\big|_{\Omega(S)}$ and continuous functions of S with respect to the C^1-topology of embeddings.*

(2) *If $\phi^1_{H^\alpha}(o_M) = \phi^1_{H^\beta}(o_M)$, then*

$$\rho(H^\alpha, S) - \rho(H^\beta, S) = c(H^\alpha, H^\beta),$$

where $c(H^\alpha, H^\beta)$ does not depend on the choice of $S \subset M$.

(3) *When $H \equiv 0$, $\rho(H, S) = 0$ for all $S \subset M$.*

(4) *We have*

$$\int_0^1 -\max_x\,(H^\beta - H^\alpha)dt \le \rho(H^\beta, S) - \rho(H^\alpha, S) \le \int_0^1 -\min_x\,(H^\beta - H^\alpha)dt.$$

In particular,

$$\int_0^1 -\max_x H dt \le \rho(H, S) \le \int_0^1 -\min_x H dt.$$

(5) $|\rho(H^\beta, S) - \rho(H^\beta, S)| \leq \|H^\beta - H^\alpha\|_{C^0}$ *which in particular implies that for each* $S \subset M$, *the function* $H \mapsto \rho(H, S)$ *is continuous with respect to the topology induced by the* C^0*-norm of* H.

Two special cases are worthwhile to mention here: When $S = \{pt\}$, we define a function, for each $q \in M$,

$$f_H(q) := \rho(H, \{q\}),$$

which becomes a continuous function on M and which we call the *basic phase function* of H (or $L = \phi_H^1(o_M)$). This function has the following remarkable property.

Theorem III. (1) *The basic phase function* f_H *depends only on* $L = \phi_H^1(o_M)$ *(up to addition of constant) and is smooth away from a set of co-dimension at least one and satisfies*

$$\text{osc}(f_H) := \max f_H - \min f_H \leq \|H\|, \tag{1.7}$$

where $\|H\|$ *is the Hofer's norm.*

(2) *At smooth points of* q, *it satisfies*

$$(q, df_H(q)) \in L = \phi_H^1(o_M).$$

In other words, the graph $G_{f_H} \subset M \times \mathbb{R}$ *of* f_H *is a subset of the wave front set of* L *(independent of* H*).*

The existence of such a graph part in the wave front set of L was first observed by Sikorav in the theory of generating functions. One novelty of Theorem III is the *canonical choice* of such a graph. We note that in general there may be more than one graph parts in the wave front. The graph part given in Theorem III carries in its definition some geometric information which is to be carefully studied in the future.

If we define the Hofer's distance between Lagrangian submanifolds (Hamiltonian isotopic to each other) by

$$d(L_1, L_2) = \inf_{H : \phi_H(L_1) = L_2} \|H\|, \tag{1.8}$$

(1.7) implies, by taking infimum over $H \mapsto L$,

$$\text{osc}(f_L) := \text{osc}(f_H) \leq d(L, o_M).$$

Combining Theorem III (2), (1.7) and the fact that only the zero section contains the graph of a constant function in its wave front, we immediately prove

Corollary. *The distance (1.8) is nondegenerate, i.e., $d(L_1, L_2) = 0$ if and only if $L_1 = L_2$.*

After we announced the proof of this corollary, Eliashberg and Sikorav [14] told us that they had known that this nondegeneracy can also be proven by the techniques of generating functions but the proof is not as straightforward as or conceptually as simple as ours (In fact, the complete proof has not been written in the literature). One can pose the same nondegeneracy question on Lagrangian submanifolds on general (P, w). This general question, which was the motivation that initiated our research in this paper, is still open.[1] We would like to compare this question with that of the symplectic displacement (or disjunction) energy of Lagrangian submanifolds (see [45] or [9]).

In a sequel [43] to this paper, we study the case in which $S = M$ in a detailed way and construct some cohomological invariants that are closely tied to the *pants product* in Floer cohomology and the group operation on the space of Hamiltonians $H : P \times [0,1] \to \mathbb{R}$,

$$H\#K(x,t) = H(x,t) + K((\phi_H^t)^{-1}(x), t).$$

In this case, $N^*S = o_M$ and so it reduces to the situation of (1.3) but contains many more interesting geometric and algebraic structures, which we refer to [43].

Many of the previous and recent works on the Floer homology (mostly for the study of Hamiltonian diffeomorphisms not for the Lagrangian intersections, though) has provided to us much insight, and analytical and geometrical background for the present work. We would like to cite here the references that influenced us most in writing this paper: First, the paper [56] by Viterbo on the generating function approach has constantly provided the direction of our research, in which [23] has helped to formulate the definition of our invariants using the Floer theory. The paper [41] by the present author and [9] by Chekanov provided *the first applications* of the relative Floer theory to the problems of symplectic topology beyond the Arnold conjecture, which has encouraged us to look for more applications, which in turn has led to the research in this paper. Next, [25], [26] and [6] taught us the Morse-Witten theory framework on the cup product in the classical homology theory. [47]

[1]Note added in proof:
Chekanov recently proved the nondegeneracy for arbitrary compact Lagrangian submanifolds in tame symplectic manifolds in a preprint entitled "Hofer's symplectic energy and invariant metrics on the space of Lagrangian embeddings".

and [7] explained well its quantized picture, the pants product in the nonrelative Floer theory in a rigorous way. [25], [26] also described the similar picture in the relative context. We learned from the paper [9] by Chekanov an important calculation involving the action functional, from which many of the calculations we do in this paper were inspired. The papers [49], [50] contain an elegant exposition on the Maslov index, which has been useful in our solving both the grading and orientation problems. Finally, our joint paper [27] with Fukaya has provided a crucial analytical step in [43] in relation to the pants product. Of course, without Floer's pioneering works [17] - [21], the present work would not have been possible.

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Notation.

(1) $H\#K(x,t) = H(x,t) + K((\phi_H^t)^{-1}(x),t)$.

(2) $\overline{H}(x,t) = -H(\phi_H^t(x),t)$.

(3) $\widetilde{H}(x,t) = -H(x,1-t)$.

(4) $\mathcal{H}^{ac}(P) =$ the set of asymptotically constant Hamiltonians on P.

(5) $\mathcal{D}_\omega^{ac}(P) =$ the set of Hamiltonian diffeomorphisms generated by $\mathcal{H}^{ac}$.

(6) $\phi_H =$ the time-one map of the equation $\dot{z} = X_H(z)$.

(7) $H \mapsto \phi$ if and only if $\phi = \phi_H$.

(8) $o_M =$ the zero section of T^*M.

(9) $H \mapsto L$ if and only if $L = \phi_H(o_M)$.

(10) $z_H^p : [0,1] \to T^*M; \quad z_H^p(t) = \phi_H^t((\phi_H^1)^{-1}(p))$.

(11) $\Omega = \{z : [0,1] \to T^*M \mid z(0) \in o_M\}$.

(12) $N^*S =$ the conormal bundle of S.

(13) $\Omega(S) = \{z \in \Omega \mid z(1) \in N^*S\}$.

(14) $\Omega(M) = \{z : [0,1] \to T^*M \mid z(0),\, z(1) \in o_M\}$.

Conventions.

(1) The Hamiltonian vector field X_H is defined by $X_H \rfloor \omega = dH$.

(2) An almost complex structure is called *compatible* to ω if the bilinear form $\langle\ ,\ \rangle = \omega(\cdot, J\cdot)$ defines a Riemannian metric.

2. Preliminaries

In this section, we give brief descriptions of the various aspects of symplectic topology that are relevant to the results we prove in the present paper.

2.1. Hofer's geometry.

On a symplectic manifold (P, ω), we denote by $\mathcal{D}^c_\omega(P)$ the set of *Hamiltonian diffeomorphisms* which is the collection of time-one maps ϕ^1_H of the Hamiltonian equation

$$\dot{z} = X_H(z),$$

where $H : P \times [0,1] \to \mathbb{R}$ is the smooth function with compact support. We denote $\phi_H = \phi^1_H$ and by $\mathcal{H}^c$ the set of such (time dependent) Hamiltonians and denote by

$$H \mapsto \phi \tag{2.1}$$

if $\phi = \phi^1_H$, and say that H *generates* ϕ or ϕ *is generated by* H. As it will be clear in the later sections, it seems to be more natural to allow such Hamiltonians that are constant outside a compact set of P. When the manifold P has many "ends", one may even allow the Hamiltonian to have different constants on different ends. We call such H's *asymptotically constant* Hamiltonians. Although it is not essential to use this larger class of Hamiltonians in this paper, it seems most natural to look at when one studies compact symplectic manifolds with more than one boundary components or non-compact manifold with more than one ends. Because of this and some normalization that we will adapt in this paper, we use the asymptotically constant Hamiltonians instead of compactly supported ones.

We denote

$$\mathcal{H}^{ac}(P) = \{H : P \times [0,1] \to \mathbb{R} \mid H \text{ is asymptotically constant}\}.$$

For each $H \in \mathcal{H}^{ac}$, we define the *support* of H as the intersection of the complements of open sets where H is constant and denote it by Supp H. Unlikely from $\mathcal{H}^c(P)$, $\mathcal{H}^{ac}(P)$ is invariant under the addition of constants. Since the Hamiltonian vector field associated to any Hamiltonian $H \in \mathcal{H}^{ac}(P)$ still has compact support, its flow will be well-defined and so we can consider the time one map $\phi_H := \phi_H^1$ for each $H \in \mathcal{H}^{ac}(P)$. We denote

$$\mathcal{D}^{ac}_\omega(P) = \{\phi_H \in \mathrm{Symp}_\omega(P) \mid H \in \mathcal{H}^{ac}(P)\}.$$

We would like to note that if P has only one end, then the two sets $\mathcal{D}^{ac}_\omega(P)$ and $\mathcal{D}^{c}_\omega(P)$ coincide but are different in general. Just as in the case of $\mathcal{D}^{c}_\omega(P)$, it is easy to check that $\mathcal{D}^{ac}_\omega(P)$ is indeed a normal subgroup of $\mathrm{Symp}_\omega(P)$ which is nicely encoded by the following correspondences: When $H \mapsto \phi$ and $K \mapsto \psi$, we have

(2.2) • $\overline{H}_t(x) = -H(\phi_H^t(x), t) \mapsto (\phi_H^t)^{-1}$.

(2.3) • $H\#K(x,t) := H(x,t) + K((\phi_H^t)^{-1}(x), t) \mapsto \phi_H^t \circ \phi_K^t$.

(2.4) • $H(\Phi(x), t) \mapsto \Phi^{-1} \circ \phi_H^t \circ \Phi$

for *any* symplecticdiffeomorphism.

It is easy to see that the operation # provides a group structure on $\mathcal{H}^{ac}(P)$ or $\mathcal{H}^c(P)$ with respect to which the zero function plays the role of the identity and $\overline{H}$ is the inverse of H. Recall that the Lie algebra of the group $\mathcal{D}^{c}_\omega(P)$ is the set of compactly supported (time independent) Hamiltonian h's i.e., $C_c^\infty(P)$. The Lie algebra of $\mathcal{D}^{ac}_\omega(P)$ is the set of (time independent) Hamiltonian h's that are asymptotically constant and whose asymptotic values c_j's on the ends satisfy

$$c_1 + c_2 + \cdots + c_\ell = 0, \tag{2.5}$$

where ℓ is the number of the ends of P. We denote by $C_b^\infty(P)$ the set of such functions on P. Here the subscript b stands for the word "balanced". We denote the total oscillation of $h \in C_b^\infty(P)$ by

$$\mathrm{osc}(h) := \max_{x\in P} h(x) - \min_{x \in P} h(x). \tag{2.6}$$

Note that osc : $C_b^\infty(P) \to \mathbb{R}^+$ is invariant under the pull-back operation by diffeomorphisms on P. Now, Hofer's norm of $H \in \mathcal{H}^{ac}$ is defined to be

$$\begin{aligned} \|H\| &:= \int_0^1 \operatorname{osc}(H_t)dt = \int_0^1 \max H_t - \min H_t dt \\ &= \int_0^1 \operatorname{osc}(H_t \circ \phi_H^t)dt, \end{aligned} \tag{2.7}$$

which is just the length of the path $t \mapsto \phi_H^t$ measured by the right-invariant Finsler structure on $\mathcal{D}_w^{ac}(P)$ given by (2.6) at $\phi =$ id. Then we define the group norm $\|\phi\|$, which corresponds to the distance from the identity to ϕ in terms of the Finsler structure by

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\| \tag{2.8}$$

for $\phi \in \mathcal{D}_\omega^{ac}(P)$. Using (2.2)–(2.4), it is easy to check that $\|\phi\|$ satisfies the identities:

(1) $\|\mathrm{id}\| = 0$.

(2) $\|\phi\psi^{-1}\| = \|\psi^{-1}\phi\|$.

(3) $\|\phi\psi\| \leq \|\phi\| + \|\psi\|$.

(4) $\|\Phi^{-1} \circ \phi \circ \Phi\| = \|\phi\|$ for any symplectic diffeomorphism Φ.

Therefore, we have a bi-invariant (pseudo)-distance, the so called *Hofer's distance* on $\mathcal{D}_\omega^{ac}(P)$ defined by

$$d(\phi, \psi) = \|\phi^{-1}\psi\|. \tag{2.9}$$

It is a highly nontrivial fact to prove that this pseudo-distance on $\mathcal{D}_\omega^c(P)$ is indeed a distance or equivalently that the (pseudo)-norm in (2.8) is nondegenerate. The same applies to $\mathcal{D}_\omega^{ac}(P)$. In fact, Lalonde-McDuff [32] proves that this fact is equivalent to the non-squeezing theorem on $B^2(r) \times P$.

Theorem 2.1 [Hofer ($\mathbb{R}^{2n}$), Lalonde-McDuff (in general)]. *The norm defined as in (2.8) is non-degenerate, i.e., $\|\phi\| = 0$ if and only if $\phi =$ id.*

Hofer's proof [30] uses a variational theory of the action functional

$$\mathcal{A}_H(\gamma) = \int_\gamma pdq - \int H(\gamma(t), t)dt$$

on the Sobolev space $H^{1/2}(S^1, \mathbb{C}^n)$, while Lalonde-McDuff [32] showed that this theorem is a consequence of a general non-squeezing theorem, through an ingenious way of constructing optimal embedding of balls: *"For any symplectic manifold (P^{2n}, w), the standard ball $B^{2(n+1)}(R) \subset \mathbb{C}^{n+1}$ can be symplectically embedded into $(B^2(r) \times P, w_0 \oplus w)$ only if $R \leq r$"*. After then, they used the theory of pseudo-holomorphic curves to prove the non-squeezing theorem, together with an ingenious *"wrapping construction"* of symplectic balls they call.

We now develop the analogue of the Hofer's geometry on the space of Lagrangian submanifolds. As in the case of diffeomorphisms, we take the point of view of Finsler geometry. Let L_0 be a fixed compact Lagrangian submanifold in (P, w) and denote by $\Lambda_\omega(L_0 : P)$ the set of Lagrangian submanifolds Hamiltonian isotopic to L_0. The tangent space of $\Lambda_\omega(L_0 : P)$ at $L \in \Lambda_\omega(L_0 : P)$ can be canonically identified with

$$\widetilde{C}^\infty(L) := C^\infty(L)/\{\text{constant functions on } L\}$$

via

$$f \mapsto \xi_f := \widetilde{w}^{-1}\big|_{T^*L}(df),$$
$$\widetilde{C}^\infty(L) \to \Lambda(NL),$$

where NL is the normal bundle of L in P. We now define a norm on this set by

$$\operatorname{osc}_L(f) := \max_{x\in L} f - \min_{x \in L} f \tag{2.10}$$

for $f \in \widetilde{C}^\infty(L)$, and a length of the Hamiltonian isotopy $s \mapsto \overline{L} = \{L_s\}$ between L_α and L_β by

$$\|\overline{L}\| = \int_0^1 \operatorname{osc}_{L_s}(f_s)ds = \int_0^1 \operatorname{osc}_L(f_s \circ (\phi_H^s)^{-1})ds,$$

where $f_s \in \widetilde{C}^\infty(L_s)$ is the element corresponding to the tangent vector at L_s of the path $\overline{L}$, i.e.,

$$\frac{d}{ds}(L_s), \quad s \in [0, 1].$$

For given two L_α, $L_\beta \in \Lambda_\omega(L_0, P)$, we define the (pseudo)-distance by

$$d(L_\alpha, L_\beta) = \inf_\phi \|\overline{L}\|, \tag{2.11}$$

where the infimum is taken over all $\phi \in \mathcal{D}^c_\omega(P)$ with $\phi(L_1) = L_2$. It immediately follows from (2.2)–(2.4) that d is symmetric and satisfies the triangle inequality. The main non-trivial question is whether this pseudo-distance is indeed nondegenerate i.e., whether it satisfies

$$d(L_1, L_2) = 0 \text{ if and only if } L_1 = L_2. \tag{2.12}$$

Whether (2.12) holds in general is still an open question, but we will prove later, as a consequence of our construction of new symplectic invariants, that this is at least true for the case $L_0 = o_M$, $P = T^*M$ where M is any compact manifold.

Theorem 9.3. *Let $P = T^*M$, $\omega = -d\theta$ be the canonical symplectic structure and $L_0 = o_M$ the zero section of T^*M. Then the Hofer's distance defined as above is nondegenerate.*

The topology induced by this distance will be the one we will take as the topology given on the space of Lagrangian submanifolds when we consider the continuity property of various symplectic invariants; we will define in the later sections.

2.2. Generating functions and Viterbo's invariants.

For a given Lagrangian submanifold $L \subset T^*M$, we call a function $S : E \to \mathbb{R}$ a *generating function* for L if L can be expressed as

$$L = \left\{ \left(x, \frac{\partial S}{\partial x}(e)\right) \Big| \frac{\partial S}{\partial \xi}(e) = 0 \right\},$$

where the map $\pi : E \to M$ is a *submersion* (typically a vector bundle), and $\frac{\partial S}{\partial \xi}$ is the fiber derivative and $\frac{\partial S}{\partial x}(e) \in T^*_x M$ is $(T^*_x \pi)^{-1}(dS(e))$ which is well-defined since we assume $T_x\pi : T_e E \to T_x M$ is surjective. We denote the *fiber critical set* by

$$\Sigma_S = \{e \in E \mid \frac{\partial S}{\partial \xi}(e) = 0\}, \tag{2.13}$$

and by $i_S : \Sigma_S \to T^*M$ the map

$$i_S(e) = \big(x, \frac{\partial S}{\partial x}(e)\big) \quad \text{for } e \in \Sigma_S. \tag{2.14}$$

The well-known important facts on the generating function is that the map $i_S : \Sigma_S \to T^*M$ is a Lagrangian immersion and the identity

$$i_S^*\theta = d(S\big|_{\Sigma_S}) \quad \text{on } \Sigma_S \tag{2.15}$$

holds. An immediate consequence of (2.15) is that if L allows a generating function, it must be *exact.* When E is a (finite dimensional) vector bundle over M, one can introduce a special generating function called a *generating function quadratic at infinity* (abbreviated as GFQI): A generating function $S : E \to \mathbb{R}$ is called a GFQI if $S(x,\xi) - Q(x,\xi)$ has compact support where $Q(x,\xi)$ is a fiberwise nondegenerate quadratic form in ξ.

The following is the basis of the Viterbo's construction of symplectic invariants.

Theorem 2.2 [Laudenbach-Sikorav [34], [53], Viterbo [56]]. *If $L = \phi(o_M)$ for $\phi \in \mathcal{D}^c_\omega(T^*M)$, then L has a GFQI. Moreover it is essentially unique up the stabilization and the fiber preserving diffeomorphisms.*

This has the consequence that the cohomology group $H^*(S^b, S^a)$ is independent of the choice of S but depends only on L *if one normalizes S appropriately.* Note that for $c > 0$ sufficiently large, we have

$$(S^c, S^{-c}) = (Q^c, Q^{-c}) \simeq (D(E^-), S(E^-)),$$

and so

$$H^*(S^c, S^{-c}) \simeq H^{*-k}(M), \quad k = \dim D(E^-),$$

which is independent of S as long as c is sufficiently big. Here we denote by E^- the negative bundle of the quadratic form Q, and by $D(E^-)$ and $S(E^-)$ the disc and the sphere bundle associated to E^-. One denotes

$$E^\infty = Q^c, \quad E^{-\infty} = Q^{-c}$$

for any such c. Now the Thom isomorphism provides the isomorphism

$$H^*(M) \to H^*(D(E^-), S(E^-)) \cong H^*(E^\infty, E^{-\infty}),\ u \mapsto Tu := \pi^*u \cup \mathcal{T}_{E^-}$$

where $\mathcal{T}_{E^-}$ is the Thom class of the vector bundle E^-.

Definition 2.3. Let S be a GFQI for $L = \phi(o_M) \subset T^*M$. For each $u \in H^*(M,\mathbb{R})$, we assign the number

$$c(S,u) := \inf_\lambda \{\lambda \mid j_\lambda^* Tu \neq 0 \ \text{ in } H^*(E^\lambda, E^{-\infty})\}.$$

Under a suitable normalization, one proves that $c(S,u)$ does not depend on S but only on L as long as S generates L. Therefore, one could define $c(L,u)$, as invariants of L, by the common number

$$c(L,u) := c(S,u)$$

for *suitably normalized* S's. However it is not completely clear what would be the best normalization in general. To define invariants of compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$, Viterbo uses a compactification of Graph $\phi \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ which provides a natural normalization in this case. (See [56] for details.) One of the main theorems in [56] we would like to mention is the following.

Theorem 2.4 [Viterbo [56]]. *Set* $\gamma(L) = c(L, \mu_M) - c(L,1)$ *where* $1 \in H(M,\mathbb{R})$, $\mu_M \in H^n(M,\mathbb{R})$ *are the canonical generators respectively. Then we have* $\gamma(L) \geq 0$ *and*

$$\gamma(L) = 0 \quad \textit{if and only if } L = o_M.$$

2.3. Action functional: the canonical generating function.

For a notational convenience, we adopt the notation

$$H \mapsto L \quad \text{if } L = \phi_H^1(o_M). \tag{2.16}$$

When $H \mapsto L$ is given, we consider the classical action functional

$$\mathcal{A}_H(\gamma) = \int \gamma^*\theta - \int_0^1 H(\gamma(t),t)\,dt$$

on the space of paths *free at its final time*

$$\Omega = \{\gamma : I \to T^*M \mid \gamma(0) \in o_M\}.$$

The space Ω has the natural structure of the fiber bundle

$$p : \Omega \to M, \quad p(\gamma) := \pi(\gamma(1)),$$

where $\pi : T^*M \to M$ is the canonical projection. We denote its fiber at $q \in M$ by Ω_q, i.e.,

$$\begin{aligned}\Omega_q :=& \{\gamma \in \Omega \mid \gamma(1) \in T_q^*M\}\\ =& \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in o_M,\ \gamma(1) \in T_q^*M\}.\end{aligned}$$

We recall the first variation formula of $\mathcal{A}_H$ for general path γ,

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 (\omega(\dot\gamma,\xi) - dH(\gamma)\xi) - \langle \xi(0), \theta(\gamma(0))\rangle + \langle \xi(1), \theta(\gamma(1))\rangle, \tag{2.17}$$

where $\langle v, \theta(p)\rangle$ is the pairing $\theta(p)(v)$. This can be derived by a direct computation. When we restrict to $\gamma \in \Omega$ i.e., $\gamma(0) \in o_M$, (2.17) becomes

$$d\mathcal{A}_H(\gamma)\xi = \int_0^1 (\omega(\dot\gamma,\xi) - dH(\gamma)\xi) + \langle \xi(1), \theta(\gamma(1))\rangle \tag{2.18}$$

since $\theta|_{o_M} \equiv 0$. From this, we see that the *fiber derivative* $\frac{d\mathcal{A}_H}{d\xi}$ satisfies that $\frac{d\mathcal{A}_h}{d\xi}(\gamma) = 0$ if and only if

$$\dot\gamma \rfloor w - dH_t(\gamma) = 0, \text{ i.e., } \dot\gamma = X_H(\gamma). \tag{2.19}$$

In other words, we have the *fiber critical set* of $\mathcal{A}_H$

$$\begin{aligned} \Sigma_{\mathcal{A}_H} &= \{\gamma \in \Omega \mid \dot\gamma = X_H(\gamma)\} \\ &= \{\gamma \in \Omega \mid \gamma(t) = \phi_H^t(\phi_H^1)^{-1}(p),\ p \in L = \phi_H^1(o_M)\}. \end{aligned} \tag{2.20}$$

Furthermore, it follows from (2.19) that the map

$$i_{\mathcal{A}_H} : \Sigma_{\mathcal{A}_H} \to T^*M \quad \text{defined as in (2.14)}$$

is nothing but

$$i_{\mathcal{A}_H}(\gamma) = \gamma(1) = \phi_H^1(\gamma(0)). \tag{2.21}$$

Now, (2.18)–(2.21) precisely mean that $\mathcal{A}_H : \Omega \to \mathbb{R}$ is a *generating function* of $L = \phi_H^1(o_M)$.

¿From the description (2.20) of $\Sigma_{\mathcal{A}_H}$, one can associate a number $\mathcal{A}_H(z_H^p)$ to each $p \in L$ where

$$z_H^p(t) = \phi_H^t((\phi_H^1)^{-1}(p)). \tag{2.22}$$

We will adopt this definition for z_H^p throughout this paper.

Definition 2.5. **[Action spectrum of H].** We define for each $H \in \mathcal{H}^{ac}$

$$\mathrm{Spec}(H) := \{\mathcal{A}_H(z_H^p) \mid p \in L = \phi_H^1(o_M)\}$$

and call it *the action spectrum* of H.

The following proposition shows that $\mathrm{Spec}(H)$ depends only on L up to addition of a constant, as long as $H \mapsto L$.

Proposition 2.6. *Assume that M is connected and $H, K \in \mathcal{H}^{ac}$.*

(1) *If $H, K \mapsto L$ i.e., $L = \phi_H^1(o_M) = \phi_K^1(o_M)$, then*

$$\mathcal{A}_H(z_H^p) - \mathcal{A}_K(z_K^p) = c(H, K) \tag{2.23}$$

for all all $p \in L$. Furthermore, for a constant c_0

$$\mathcal{A}_{H+c_0}(z_{H+c_0}^p) - \mathcal{A}_H(z_H^p) = c_0 \tag{2.24}$$

for all $p \in L$. Therefore by adding an appropriate constant, we can assume

$$\mathcal{A}_H(z_H^p) = \mathcal{A}_K(z_K^p) \quad \text{for all } p \in L$$

as long as $H, K \mapsto L$.

(2) *For each $H \in \mathcal{H}^{ac}$ consider the subset of $M \times \mathbb{R}$*

$$W_H = \{(q, r) \mid q = \pi(p),\ r = \mathcal{A}_H(z_H^p),\ p \in L\}.$$

Then W_H is a wave front set of $L = \phi_H^1(o_M)$.

Proof of (1). (2.24) follows immediately from the fact

$$z_{H+c_0}^p = z_H^p$$

and the definition of $\mathcal{A}_H$. Therefore we prove only (2.23). Since L is a smooth manifold, it is obvious that the function

$$p \mapsto \mathcal{A}_H(z_H^p) - \mathcal{A}_K(z_K^p) := g(p)$$

on L is a smooth function. Therefore it is enough to show that the function g is locally constant since we assume that M and so L is connected. Therefore we compute its derivative. For each $v \in T_pL$, $p \in L$,

$$dg(p)(v) = d\mathcal{A}_H(z_H^p)(\xi_H^v) - d\mathcal{A}_K(z_K^p)(\xi_K^v)$$

where ξ_H^v and ξ_K^v are the variations of z_H^p and z_K^p respectively induced by $v \in T_pL$. More explicitly, we have

$$\xi_H^v(t) = T\phi_H^t(T\phi_H^{-1}(v)),$$
$$\xi_K^v(t) = T\phi_K^t(T\phi_K^{-1}(v)).$$

Since we assume $v \in T_pL$ and $L = \phi_H(o_M) = \phi_K(o_M)$, we have

$$\xi_H^v(1) = \xi_K^v(1) = v.$$

By the variation formula (2.18), we obtain

$$\begin{aligned} d\mathcal{A}_H(z_H^p)(\xi_H^v) &= \langle \xi_H^v(1), \theta(z_H^p(1))\rangle = \langle v, \theta(p)\rangle, \\ d\mathcal{A}_K(z_K^p)(\xi_K^v) &= \langle \xi_K^v(1), \theta(z_K^p(1))\rangle = \langle v, \theta(p)\rangle, \end{aligned} \tag{2.25}$$

and hence $dg(p)(v) = 0$ for all $v \in T_pL$ i.e., $dg = 0$. This finishes the proof of (1). q.e.d.

Proof of (2). Before proving Proposition 2.6 (2), we recall the definition of *exact* Lagrangian submanifolds: If $L \subset T^*M$ is an exact Lagrangian submanifold, then $i^*\theta$ is exact, i.e., $\lambda^*\theta = df$ for some function f on L. The wave front of L is just the projection of the Legendrian lift

$$\widetilde{L} = \{(p, r) \in T^*M \times \mathbb{R} \mid r = f(p), p \in L\}$$

to $M \times \mathbb{R}$ by the map $(\pi \times \text{ id}) : T^*M \times \mathbb{R} \to M \times \mathbb{R}$. The projection is nothing but

$$W_f := \{(q, r) \in M \times \mathbb{R} \mid q = \pi(p), r = f(p), p \in L\}.$$

Now if we restrict to the case $L = \phi_H^1(o_M)$, the formula (2.25) explicitly shows that one can take the function f defined by

$$f(p) := \mathcal{A}_H(z_H^p), \quad z_H^p(t) := \phi_H^t(\phi_H^1)^{-1}(p)$$

for $p \in L$. Hence the proof. q.e.d.

Note that if $H \in \mathcal{H}^{ac}$, then $H + c_0$ is also in $\mathcal{H}^{ac}$ which makes $\mathcal{H}^{ac}$ more natural to consider than $\mathcal{H}^c$.

Now we study the size of the set of critical values of $\mathcal{A}_H|_{\Omega(S)}$ as a subset of $\mathbb{R}$, which is in general useful to prove the invariance property of the symplectic invariants that we shall define later. Similar results were used before in the study of periodic orbits and the associated invariants of Hamiltonian diffeomorphisms (see e.g., [31]).

Proposition 2.7. *For each submanifold $S \subset M$, the set of critical values of $\mathcal{A}_H|_{\Omega(S)}$ is a compact nowhere dense subset of $\mathbb{R}$. We denote*

$$\begin{aligned} \operatorname{Spec}(H, S) &= \text{the set of critical values of } \mathcal{A}_H|_{\Omega(S)} \\ &= \{\mathcal{A}_H(z_H^p) \mid p \in \phi_H(o_M) \cap N^*S\}. \end{aligned}$$

Proof . Consider the function $h : N^*S \to \mathbb{R}$ defined by

$$h(p) := \mathcal{A}_H(z_H^p) \quad \text{for} \quad p \in N^*S.$$

¿From the definitions of z_H^p and $\mathcal{A}_H$, it follows that h is a smooth function defined on N^*S which is a *finite dimensional* manifold. By the variational formula (2.17), we get

$$dh(p)(v) = -\langle \theta(z_H^p(0)), T\phi_H^{-1}(v)\rangle + \langle \theta(p), v\rangle = -\langle \theta(z_H^p(0)), T\phi_H^{-1}(v)\rangle$$

for all $v \in T_p(N^*S)$. If $p \in \phi_H(o_M) \cap N^*S$, then $z_H^p(0) \in o_M$ which implies that $\theta(z_H^p(0)) = 0$ so that $dh(p) = 0$. Therefore all the points $p \in \phi_H(o_M) \cap N^*S$ are critical points of h, and the corresponding critical values are $\mathcal{A}_H(z_H^p)$. Hence we have shown that

$$\mathrm{Spec}(H, S) \subset \text{the set of critical values of } h.$$

However by the (classical) Sard's theorem, the set of critical values of h is of measure zero and therefore nowhere dense, and so is $\mathrm{Spec}(H, S)$. The compactness of $\mathrm{Spec}(H, S)$ immediately follows from that it is a closed subset of $\mathrm{Spec}(H)$ which is compact. Note that $\mathrm{Spec}(H)$ is compact because we assume that M is compact and so the wave front set of $L = \phi_H(o_M)$ is compact for any H. q.e.d.

For the later purposes, it is important to understand the relation between $\mathcal{A}_H$ on $\Omega(S)$ and the Floer's action functional $\underline{a}^S$ on the space

$$\Omega(L_0, L_1) = \{\gamma : [0, 1] \to T^*M \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}$$

when

$$L_0 = \phi_H(o_M), \; L_1 = N^*S.$$

The crucial property which the functional $\underline{a}^S$ must have is that its gradient flow in terms of certain L^2-type metrics on $\Omega(L_0, L_1)$ becomes the Cauchy-Riemann equation with Lagrangian boundary condition as in (1.1). We recall how the functional $\underline{a}^S$ is defined in the case where $L_1 = \phi(L_0)$ in [18]: Fix an intersection point $p_0 \in L_0 \cap L_1$. For each given $\gamma \in \Omega(L_0, L_1)$,

FIGURE 1

we pick a surface Γ drawn as above (such a surface exists if one restricts to a certain component of $\Omega(L_0, L_1)$), and define

$$\underline{a}^S(\gamma) = \int_\Gamma w. \tag{2.26}$$

However in our case $L_0 = \phi_H(o_M)$ and $L_1 = N^*S$ as above, this definition does not work in general by two reasons: First, there may not exist the bounding surface Γ and secondly, even if there is, the value in (2.26) may depend on the choice of Γ. We refer readers to [38] for similar discussions concerning this point. So we will define the functional $\underline{a}^S$ directly using the canonical one form θ on T^*M and the fact that $\theta|_{N^*S} \equiv 0$ and L_0 is an *exact* Lagrangian submanifold. We choose $f_{L_0} : L_0 \to \mathbb{R}$ such that

$$df_{L_0} = \theta|_{L_0}, \tag{2.27}$$

and define

$$\underline{a}^S(\gamma) = \int \gamma^*\theta + f_{L_0}(\gamma(0)) \tag{2.28}$$

on the space $\Omega(L_0, L_1)$. Using the variation formula (2.17), we compute for $\xi \in T_\gamma(\Omega(L_0, L_1))$

$$\begin{aligned} d\underline{a}^S(\gamma)(\xi) &= \int (\omega(\dot\gamma, \xi))dt - \langle \xi(0), \theta(\gamma(0))\rangle + \langle df_{L_0}(\gamma(0)), \xi(0)\rangle \\ &= \int_0^1 (\omega(\dot\gamma, \xi))dt, \end{aligned}$$

where the second equality comes from (2.27). Therefore the derivative $\underline{a}^S$ will be the same as that of the Floer's original functional if it exists. We will still call $\underline{a}^S$ as above the Floer's action functional.

Since f_{L_0} is defined up to addition of a constant, so is $\underline{a}^S$. Now we have the following important fact.

Proposition 2.8. *Let p_1 and $p_2 \in \phi_H(o_M) \cap N^*S$, and consider them as constant paths in $\Omega(L_0, L_1)$. Let $z_H^{p_i}$, $i = 1, 2$ be the elements in $\Omega(M)$ defined as before, i.e.,*

$$z_H^{p_i}(t) = \phi_H^t(\phi_H^{-1}(p_i)) \quad \textit{for } i = 1, 2.$$

Then we have

$$\mathcal{A}_H(z_H^{p_2}) - \mathcal{A}_H(z_H^{p_1}) = \underline{a}^S(p_2) - \underline{a}^S(p_1). \tag{2.29}$$

By taking the function $f_{L_0} = \mathcal{A}_H(z_H^p)$ for $p \in L_0 = \phi_H(o_M)$, we may assume that

$$\mathcal{A}_H(z_H^p) = \underline{a}^S(p) \quad \textit{for all} \;\; p \in \phi_H(o_M) \cap N^*S. \tag{2.30}$$

Proof . We pick any curve γ_1 in $\phi_H(o_M)$ with $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Then combining (2.25), (2.27) and that $p_1, p_2 \in \Omega(L_0, L_1)$ are considered to be constant paths, we have

$$\begin{aligned} \mathcal{A}_H(z_H^{p_2}) - \mathcal{A}_H(z_H^{p_1}) &= \int_{\gamma_1} \theta = \int_{\gamma_1} df_{L_0} \\ &= f_{L_0}(p_2) - f_{L_0}(p_1) = \underline{a}^S(p_2) - \underline{a}^S(p_1), \end{aligned} \tag{2.31}$$

which finishes the proof. q.e.d.

2.4. Semi-infinite cycles.

The action functional $\mathcal{A}_H$ on the space

$$\Omega = \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in o_M\}$$

as a generating function has an advantage over its finite dimensional version in that it is canonical and does not involve any nontrivial choice. This enables us to capture the geometric insight of the symplectic invariants which we are going to construct, via the Floer theory, using the action functional.

Recall from [56] and [55] that the proof of the uniqueness of GFQI of a given Lagrangian submanifold $L = \phi_H(o_M)$ up to the *stabilization* and the *gauge invariance* forms one of the crucial ingredient in Viterbo's construction and requires some sophisticated topological machinery [56]. Of course, we have to pay off: the action functional is defined on the *infinite dimensional* path space Ω, for which it is well known (see e.g., [29]) that the classical critical point theory on T^*M has serious limitations in general cases other than when $M = \mathbb{R}^n$, i.e., $T^*M = \mathbb{R}^{2n} \simeq \mathbb{C}^n$. Still, however, it is very natural and conceptually simpler to develop analogues to Viterbo's work [56] directly working with the action functional $\mathcal{A}_H : \Omega \to \mathbb{R}$. This attempt could be considered as the *direct approach* against the *finite dimensional approximation* in the critical point theory of the action functional. There are two major difficulties to overcome in this attempt:

(1) *The standard direct approach to the functional $\mathcal{A}_H : \Omega \to \mathbb{R}$ for general M does not work by various reasons (e.g., lack of the global coordinates, the failure of Palais-Smale conditions and etc.)* – Analytical aspect –

(2) *There does not exist the Thom isomorphism on the fibration $p : \Omega \to M$ in the classical algebraic topological sense, because the fiber Ω_q is infinite dimensional. More geometrically saying, it is not a priori obvious which mini-maxing sets one should choose to have the linking properties and so to pick out certain critical values of $\mathcal{A}_H$.* – Topological aspect –

We will overcome these difficulties simultaneously via versions of the *Floer theory* of Lagrangian submanifolds. We would like to mention that Floer himself invented the Floer homology in the precisely same kind of reasons.

The two difficulties mentioned here turn out to be inter-related. In the classical critical point theory, the *mini-maxing sets* are the ones that define *nontrivial cycles in terms of the gradient flow of the given functional*. In the literature [48], [5], [4] and so on, the choice of such cycles depend on the type of the given functional. Because the classical action functional is so called *strongly indefinite*, the notion of *semi-infinite cycles* has been implicitly used in the literature related to the periodic orbit problem of the Hamiltonian system on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. Mostly in the traditional direct approach, the global gradient flow of the action functional on the Sobolev space $H^{1/2}(S^1, \mathbb{C}^n)$ is well-defined and satisfies versions of Palais-Smale condition, and so one can apply the classical variational theory using the mountain-pass type lemma. With these experiences at hand, we will try to choose our *semi-infinite cycles* with respect to which the Floer theory on Ω works well.

We start with the formula (2.18):

$$d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 (\omega(\dot\gamma, \xi) - dH_t(\gamma)\xi)dt + \langle \xi(1), \theta(\gamma(1))\rangle.$$

As usual, we would like to write down the gradient flow of $\mathcal{A}_H$ on Ω with respect to some metric on Ω. We choose an almost complex structure J on T^*M that is *compatible* with the symplectic structure: We say that J is compatible to ω, if the bilinear form $\langle \ , \ \rangle$ defined by

$$\langle \ , \ \rangle_J := \omega(\cdot, J\cdot) \tag{2.32}$$

defines a Riemannian metric. Then one can re-write (2.18) as

$$d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \langle J(\dot\gamma - X_{H_t}(\gamma)), \xi\rangle_J dt + \langle \xi(1), \theta(\gamma(1))\rangle. \tag{2.33}$$

When one tries to write down the equation corresponding to the gradient flow of $-\mathcal{A}_H$ as in the Floer theory, one immediately encounters a difficulty due to the boundary term $\langle \xi(1), \theta(\gamma(1)\rangle$ in (2.33). To do the Floer theory correctly in *the analytical point of view*, one should try to get rid of the difficulty *by choosing certain subset of* Ω *so that if we restrict the functional* $\mathcal{A}_H$ *thereto, the boundary term drops out for the gradient flow of the restricted function.* It is a remarkable fact that this attempt of ours to overcome *the analytical difficulty* gives rise to the way to associating a semi-infinite cycle to *each* compact submanifold of M and hence solves the *topological difficulty* mentioned above as well.

¿From the definition of the canonical one-form θ on T^*M, we can re-write the boundary term as

$$\langle \xi(1), \theta(\gamma(1))\rangle = \langle T\pi\xi(1), \gamma(1)\rangle.$$

Main Observation. *The term* $\langle T\pi\xi(1), \gamma(1)\rangle$ *vanishes if one imposes the condition that* $\gamma(1)$ *lies in the co-normal bundle* $N^*S \subset T^*M$ *of any submanifold* $S \subset M$ *and* $\xi(1)$ *is tangent to* N^*S*, because* $\theta|_{N^*S} \equiv 0$ *for any submanifold* $S \subset M$*.*

We now assign to each compact submanifold $S \subset M$ a semi-infinite cycle in Ω which is *linked to*, in terms of the gradient flow of the action functional $\mathcal{A}_H$, the *fundamental cycle* $\Omega(M)$ defined by

$$\Omega(M) = \{\gamma : [0,1] \to T^*M \mid \gamma(0), \gamma(1) \in o_M \ \},$$

and in particular which cannot be pushed away to the infinity by the gradient flow of $\mathcal{A}_H$. This linking property, however, will be detected by the *Floer homology theory.* We would like to emphasize that this choice of the semi-infinite cycles does not depend on the Hamiltonian H at all as long as H is asymptotically constant, which will be a crucial ingredient in defining our symplectic invariants of Lagrangian submanifolds in the later sections.

Example 2.9.

(1) When $S = \{pt\}$, we assign to each $q \in M$ the cycle

$$\Omega\{q\} := \{\gamma \in \Omega \mid \gamma(1) \in T_q^*M\} = \Omega_q.$$

Here we have used the fact that $N^*\{q\} = T_q^*M$.

(2) When $S = M$, we have $N^*S = o_M$ and so the corresponding cycle is the fundamental cycle

$$\Omega(M) = \{\gamma \in \Omega \mid \gamma(1) \in o_M\}.$$

In the rest of this paper, we will first develop the Floer theory to each submanifold $S \subset M$ and then construct certain symplectic invariants of Lagrangian submanifolds associated to each S. Although the basic construction of the Floer homology is standard, the construction of symplectic invariants using the Floer homology is new. In the course of doing these, we discover many new aspects in the Floer theory itself and so lay the foundation for serious applications of the Floer theory to the questions of symplectic topology. We refer readers to [43] for further results in this aspect.

3. A C^0-estimate

We will concern the gradient flow of $\mathcal{A}_H\big|_{\Omega(S)}$ with respect to certain L^2-type metrics on $\Omega(S)$ for the given compact submanifold $S \subset M$. It turns out to be very important to vary the metrics on $\Omega(S)$ suitably depending on circumstances given, and so we first describe the class of the metrics on $\Omega(S)$ in detail which we are going to use.

We first note that if a Riemannian metric g is given to M, the associated Levi-Civita connection induces a natural almost complex structure on T^*M, which we denote by J_g and which we call the *canonical almost complex structure* (in terms of the metric g on M). We are going to fix the Riemannian metric g on M once for all. This canonical almost complex structure has the following properties:

(1) J_g is compatible to the canonical symplectic structure w on T^*M.

(2) For every $(q,p) \in T^*M$, J_g maps the vertical tangent vectors to horizontal vectors with respect to the Levi-Civita connection of g.

(3) On the zero section $o_M \subset T^*M$, J_g assigns to each $v \in T_qM \subset T_{(q,0)}(T^*M)$ the cotangent vector $J_g(v) = g(v,\cdot) \in T_q^*M \subset T_{(q,0)}(T^*M)$. Here we use the canonical splitting

$$T_{(q,0)}(T^*M) = T_qM \oplus T_q^*M.$$

We consider the class of compatible almost complex structures J on T^*M such that

$$J \equiv J_g \text{ outside a compact set in } T^*M,$$

and denote by j^c the class

$$j^c := \{J \mid J \text{ is compatible to } \omega \text{ and } J \equiv J_g \text{ outside a compact subset in } T^*M\}.$$

We define the *support* of J and denote

$$\text{Supp } J := \text{ the closure of } \{x \in T^*M \mid J(x) \neq J_g(x)\}.$$

The main objects that we need in defining the metrics on Ω is the following.

Definition 3.1. Let j^c be as above. We define

$$\mathcal{J}^c := \{J : [0,1] \to j^c \mid J = \{J_t\}_{0 \le t \le 1} \text{ is a smooth path}\}.$$

Each given $J \in \mathcal{J}^c$ induces a smooth path of Riemannian metrics $g_{J_t} := \omega(\cdot, J_t\cdot)$ on T^*M. We denote the corresponding norm by $|\cdot|_{J_t}$ on $T(T^*M)$. Using this, we define a metric on the space of paths in T^*M. Let $\gamma : [0,1] \to T^*M$ be a path and ξ, η be vector fields along γ. Define the inner product $\langle\langle \xi, \eta \rangle\rangle_J$ by

$$\langle\langle \xi, \eta \rangle\rangle_J := \int_0^1 \langle \xi(t), \eta(t) \rangle_{J_t} dt \tag{3.1}$$

and the associated norm by

$$\|\xi\|_J^2 := \int_0^1 |\xi(t)|_{J_t}^2 dt. \tag{3.2}$$

Note that these are the inner products and norms of L^2-type, i.e., do not involve derivatives of ξ. Using these, one can rewrite $d\mathcal{A}_H$ on Ω as

$$\begin{aligned} d\mathcal{A}_H(\gamma)(\xi) &= \int_0^1 (\omega(\dot\gamma, \xi) - dH_t(\gamma)\xi) + \langle \xi(1), \theta(\gamma(1))\rangle \\ &= \int_0^1 \langle J_t(\dot\gamma(t) - X_{H_t}(\gamma(t))), \xi(t)\rangle_{J_t} + \langle \xi(1), \theta(\gamma(1))\rangle \\ &= \langle\langle\, J(\dot\gamma - X_H(\gamma)), \xi\,\rangle\rangle_J + \langle \xi(1), \theta(\gamma(1))\rangle. \end{aligned} \tag{3.3}$$

Since we already mentioned that $\langle \xi(1), \theta(\gamma(1)\rangle$ drops out when we consider the (negative) gradient flow of $\mathcal{A}_H\big|_{\Omega(S)}$ on the space

$$\Omega(S) = \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in o_M, \gamma(1) \in N^*S\},$$

the negative gradient flow of $\mathcal{A}_H\big|_{\Omega(S)}$ in terms of the metric $\langle\langle\ ,\ \rangle\rangle_J$ on $\Omega(S)$ satisfies the equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\big(\frac{\partial u}{\partial t} - X_H(u)\big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S, \end{cases} \tag{3.4}$$

which is a perturbed Cauchy-Riemann equation with *Lagrangian boundary conditions*. Here we should emphasize that J *depends on time* in general and N^*S is *not compact*. Now all the necessary Fredholm property and compactness properties used in [17], [18] for the case of *compact* Lagrangian submanifolds will apply to (3.4), *provided we establish certain C^0-estimates for (3.4)*. This C^0-estimate is the first essential step for the Floer theory on *noncompact* symplectic manifolds. (See [23] for such an estimate for the periodic orbit problem, and [13] or [44] for the Lagrangian intersection). For the later purpose, we also have to consider the parametrized versions of (3.4). More generally, consider

$$L : \mathbb{R} \times [0,1] \times T^*M \to \mathbb{R},$$

that is smooth and such that there exists a suitable $K > 0$ such that

$$\begin{cases} L(\tau, t, u) = H^\alpha(t, u) & \text{for } \tau \le -K, \\ L(\tau, t, u) = H^\beta(t, u) & \text{for } \tau \ge K, \end{cases} \tag{3.5}$$

where $H^\alpha, H^\beta \in \mathcal{H}^{ac}$. Furthermore $\overline{J}$ is a smooth family

$$\overline{J} : \mathbb{R} \times [0,1] \to j^c$$

satisfying

$$\begin{cases} \overline{J}(\tau,t,u) = J_g & \text{outside a compact set of } T^*M, \\ \overline{J}(\tau,t,u) = J^\alpha(t,u) & \text{for } \tau \le -K, \\ \overline{J}(\tau,t,u) = J^\beta(t,u) & \text{for } \tau \ge K. \end{cases}$$

Finally consider an isotopy of submanifolds $\overline{S} = \{S_\tau\}$ such that

$$\begin{aligned} S_\tau &= S^\alpha \qquad \text{for } \tau \le -K, \\ &= S^\beta \qquad \text{for } \tau \ge K. \end{aligned}$$

Then we consider a smooth solution $u : \mathbb{R} \times [0,1] \to T^*M$ of

$$\begin{cases} \frac{\partial u}{\partial \tau} + \overline{J}\big(\frac{\partial u}{\partial t} - X_L(u)\big) = 0, \\ u(\tau,0) \in o_M, \\ u(\tau,1) \in N^*S_\tau. \end{cases} \tag{3.6}$$

The following is the main theorem in this section.

Theorem 3.2. *Assume that* $\overline{J}, L$ *and* $\overline{S}$ *as above. Then there exists a constant* $c = c(L, \overline{J}, \overline{S}) > 0$ *such that every solution* u *of* (3.6) *with*

$$\inf_{\tau\in\mathbb{R}} \mathcal{A}_{L(\tau)}(u(\tau)) > -\infty, \quad \sup_{\tau\in\mathbb{R}} \mathcal{A}_{L(\tau)}(u(\tau)) < \infty \tag{3.7}$$

satisfies

$$\sup_{(\tau,t)\in\Theta} |p(\tau,t)|_g \le c, \tag{3.8}$$

where we write $u(\tau,t) = (q(\tau,t), p(\tau,t))$ *in* T^*M, *and* $|\cdot|_g$ *is the norm on* $T^*_{q(\tau,t)}M$ *induced from the metric* g *on* M.

Proof . First, note that

$$\begin{aligned} L(\tau) &= H^\alpha \quad \text{if } \tau \le -K, \\ &= H^\beta \quad \text{if } \tau \ge K, \end{aligned}$$

and so if $\tau \le -K$, then

$$\mathcal{A}_{L(\tau)}(u(\tau)) = \mathcal{A}_{H^\alpha}(u(\tau)),$$

and hence

$$\sup_{\tau\in(-\infty,-K)} \mathcal{A}_{H^\alpha}(u(\tau)) < \infty$$

from the assumption (3.7). In particular, we have

$$\int_{-\infty}^{-K}\int_0^1 \left|\frac{\partial u}{\partial\tau}\right|^2_{J^\alpha(t)} + \left|\frac{\partial u}{\partial t} - X_{H^\alpha}(u)\right|^2_{J^\alpha(t)} dt d\tau < \infty, \tag{3.9}$$

since it is the same as

$$\mathcal{A}_{H^\alpha}(u(-\infty)) - \mathcal{A}_{H^\alpha}(u(-K)).$$

Similarly at $+\infty$ we have

$$\int_{K}^{\infty}\int_0^1 \left|\frac{\partial u}{\partial\tau}\right|^2_{J^\beta(t)} + \left|\frac{\partial u}{\partial t} - X_{H^\beta}(u)\right|^2_{J^\beta(t)} dt d\tau < \infty. \tag{3.10}$$

By the standard estimates, we can prove from (3.9) and (3.10) that

$$\lim_{\tau\to-\infty} u(\tau) = z^\alpha, \quad \lim_{\tau\to-\infty} u(\tau) = z^\beta \text{ uniformly,}$$

where z^α and z^β satisfy the equations, respectively,

$$\begin{cases} \dot z^\alpha = X_{H^\alpha}(z^\alpha), \\ z_\alpha(0) \in o_M, z_\alpha(1) \in N^*S^\alpha, \end{cases} \qquad \begin{cases} \dot z^\beta = X_{H^\beta}(z^\beta), \\ z_\beta(0) \in o_M, z_\beta(1) \in N^*S^\beta. \end{cases}$$

Then it must hold that either the $\sup_{(\tau,t)} |p(\tau,t)|_g$ where

$$u(\tau,t) = (q(\tau,\, t), p(\tau,t))$$

is realized at $\tau = \pm\infty$, or the supremum is realized at some point $(\tau_0,t_0)\in\Theta$. Since one can easily derive the C^0-estimate of the Hamiltonian paths z^α's from the assumption that L is asymptotically constant and so X_L is of compact support, in the first case we are done by the C^0-estimate of z^α's. Therefore we consider only the second case. It will be enough to prove the following.

Assertion. *If Supp* $L\cup$ *Supp* $\overline{J} \subset D_R$ *where* $D_R \subset T^*M$ *is the disc bundle*

$$D_R := \{u \in T^*M \mid |p|_g \le R\},$$

where $u(\tau,t) = (q(\tau,t), p(\tau,t))$, *then* $u(\tau_0,t_0) \in D_R$.

Suppose the contrary that $u(\tau_0, t_0) \notin D_R$, say $u(\tau_0, t_0) \in \partial D_{R'}$, $R' > R$ and that there exists an open neighborhood B_ϵ of (τ_0, t_0) in Θ for some $\epsilon > 0$ such that

$$u(B_\epsilon) \subset T^*M \backslash D_R \subset T^*M \backslash (\text{Supp } L \cup \text{ Supp } \overline{J}),$$

and so u satisfies on B_ϵ

$$\frac{\partial u}{\partial \tau} + J_g \frac{\partial u}{\partial t} = 0. \tag{3.11}$$

We consider two cases separately: the cases where $(\tau_0, t_0) \in$ Int Θ and $(\tau_0, t_0) \in \partial\Theta$. Recall that the boundary ∂D_R is of contact type, and it is J_g-convex in the sense of Gromov [28]. The following lemma is well-known e.g. is proven in [36, Lemma 2.4].

Lemma 3.3. *Let (Z, w, J) be a calibration (or an almost Kähler structure) and its boundary $\Delta = \partial Z$ be J-convex. Then no J-holomorphic curve u in Z can touch Δ at an interior point of the domain of u.*

This immediately rules out the possibility $(\tau_0, t_0) \in$ Int Θ. Now consider the case where $(\tau_0, t_0) \in \partial\Theta$, i.e., $t_0 = 1$. (The case $t_0 = 0$ is trivially removed since we assume that $u(\tau, 0) \in o_M$.) We consider the boundary curve

$$\tau \to u(\tau, 1) = (q(\tau, 1), p(\tau, 1)),$$

which becomes tangent to $\partial D_{R'} \cap N^*S$ at $(\tau_0, 1)$. Since $N^*S \cap \partial D_{R'}$ is Legendrian in $\partial D_{R'}$, the curve is tangent to the contact distribution of ∂D_r at $(\tau_0, 1)$

$$\{\xi \in T(\partial D_{R'}) \mid \xi \perp J_g \tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r} \text{ is the radial field on } T^*M\}.$$

Since u is J_g-holomorphic, u is also tangent to the contact distribution at $(\tau_0, 1)$ and in particular we have

$$\frac{\partial}{\partial t}|p(\tau, t)|^2\Big|_{(\tau_0, 1)} = 0.$$

However, this contradicts to the strong maximum principle applied to the subharmonic function (with respect to J_g)

$$(\tau, t) \mapsto |p(\tau, t)|^2 \quad \text{on } B_\epsilon(\tau_0, 1)$$

since we can assume Image $u\big|_{B_\epsilon(\tau_0,1)} \not\subset \partial D_{R'}$ by choosing ϵ slightly larger if necessary. This finishes the assertion and so the proof of Theorem 3.2. q.e.d.

4. Regular parameters

In this section, we describe the meaning of the "generic" parameters for which various moduli spaces that we are going to consider become smooth manifolds so that various versions of Floer homology which we introduce should be well-defined.

First, we introduce a subset of $\mathcal{H} = \mathcal{H}^{ac}$

$$\mathcal{H}_0 := \{H \in \mathcal{H} \mid \phi_H^1(o_M) \pitchfork o_M\}. \tag{4.1}$$

For any such Hamiltonian $H \in \mathcal{H}_0$, there are only finitely many solutions of

$$\begin{cases} \dot{z} = X_H(z), \\ z(0) \in o_M, \quad z(1) \in o_M, \end{cases} \tag{4.2}$$

i.e., critical points of $\mathcal{A}_H$ on Ω or $\mathcal{A}_H\big|_{\Omega(M)}$. For given $H \in \mathcal{H}_0$ and a compact manifold $S_0 (\subset M)$, we denote by Emb $(S_0 : M)$ the set of embeddings of S_0 into M and introduce its subset

$$\mathrm{Emb}^H = \mathrm{Emb}^H(S_0 : M) = \{S \in \ \mathrm{Emb}\ (S_0 : M) \mid N^*S \pitchfork \phi_H(o_M)\}.$$

For such $S \in$ $\mathrm{Emb}^H(S_0 : M)$, there are only finitely many solutions of

$$\begin{cases} \dot{z} = X_H(z), \\ z(0) \in o_M, \ z(1) \in N^*S, \end{cases} \tag{4.3}$$

i.e., critical points of $\mathcal{A}_H\big|_{\Omega(S)}$. We will also consider the isotopy class of a given embedding S_0 for which we denote by

$$\mathrm{Iso}\ (S_0 : M) \subset \mathrm{Emb}\ (S_0 : M)$$

and

$$\mathrm{Iso}^H = \mathrm{Iso}^H(S_0 : M) = \ \mathrm{Iso}\ (S_0 : M) \cap \ \mathrm{Emb}^H(S_0 : M).$$

By the standard transversality theorem, it follows that $\mathrm{Emb}^H(S_0 : M)$ is dense in $\mathrm{Emb}(S_0 : M)$ in the C^∞-topology. Next, we consider the regular property of the space of solutions

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S. \end{cases} \tag{4.4}$$

Then the proof of the following theorem will be standard now by combining ideas in [17], [23], [24], [40] and [43].

Theorem 4.1.

(1) *Let* $H \in \mathcal{H}_0$ *and* $S \in \mathrm{Emb}^H(S_0 : M)$. *Then there exists a dense subset* $\mathcal{J}_{H,S} \subset \mathcal{J}^c$ *such that all the solutions of (4.4) are regular, i.e., the linearization at every solution is surjective.*

(2) *Let* $S \subset M$ *and* $H \in \mathcal{H}_S := \{H \in \mathcal{H}_0 \mid \phi_H(o_M) \pitchfork N^*S\}$. *Then there exists a dense subset* $\mathcal{J}_{S,H} \subset \mathcal{J}^c$ *such that all the solutions of* (4.4) *are regular.*

We will also need the parameterized versions of this theorem.

Theorem 4.2. (1) *Let* H^α, $H^\beta \in \mathcal{H}_0$, $S \in \mathrm{Emb}^{H^\alpha} \cap \mathrm{Emb}^{H^\beta}$ *and* $J \in \mathcal{J}_{H^\alpha,S} \cap \mathcal{J}_{H^\beta,S}$. *Then there exists a dense subset of*

$$\overline{\mathcal{H}}_{H^\alpha H^\beta} := \{\overline{H} : [0,1] \to \mathcal{H} \mid H^0 = H^\alpha, H^1 = H^\beta, \overline{H} = \{H^s\}_{0 \le s \le 1}\}$$

and $K \in \mathbb{R}_+$ *such that all the solutions of*

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\big(\frac{\partial u}{\partial t} - X_{H^{\rho_K(\tau)}}(u)\big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S \end{cases} \tag{4.5}$$

are regular.

(2) *Let* S^α, $S^\beta \in \mathrm{Emb}_0^H(S_0 : M)$ *and* $J \in \mathcal{J}_{S^\alpha,H} \cap \mathcal{J}_{S^\beta,H}$. *Then there exists a dense subset of*

$$\overline{\mathrm{Emb}}_{H^\alpha H^\beta} := \big\{\overline{S} : [0,1] \to \mathrm{Emb} \mid \overline{S} = \{S_s\}_{1 \le s \le 1} \text{ a smooth isotopy with } S_0 = S^\alpha, S_1 = S^\beta\big\}$$

and $K \in \mathbb{R}_+$ *such that all the solutions of*

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\big(\frac{\partial u}{\partial t} - X_H(u)\big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S_{\rho_K(\tau)} \end{cases}$$

are regular where ρ_K *is the function as defined in Section 5.*

(3) *Let* J_α, $J_\beta \in \mathcal{J}_{H,S}$. *Then there exists a dense subset of*

$$\overline{\mathcal{J}}_{J_\alpha J_\beta} := \big\{\overline{J} : [0,1] \to \mathcal{J}^c \mid \overline{J} = \{J^s\}_{0 \le s \le 1} \text{ is smooth and } J^0 = J^\alpha,\ J^1 = J^\beta\big\}$$

and $K \in \mathbb{R}_+$ *such that all the solutions of*

$$\begin{cases} \frac{\partial u}{\partial \tau} + J_\rho\big(\frac{\partial u}{\partial t} - X_H(u)\big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S \end{cases}$$

are regular.

One could state more general version of this kind of theorem, but we state the theorems in a way that they are suitable for studying various invariance properties of the Floer homology which we define later. For the notational convenience, we denote

$$\mathcal{N}_{\mathrm{reg}}(S) = \{(H, J) \in \mathcal{H} \times \mathcal{J}^c \mid (4.4) \text{ is regular } \xi\}, \tag{4.6}$$

$$\mathcal{N}_{\mathrm{reg}}(H) = \{(S, J) \in \ \mathrm{Emb}(S_0 : M) \times \mathcal{J}^c \mid (4.4) \text{ is regular } \xi\}, \tag{4.7}$$

$$\mathcal{N}_{\mathrm{reg}} = \{(H, S) \in \mathcal{H}^{ac} \times \mathrm{Emb}\ (S_0 : M) \mid \phi_H(o_M) \pitchfork N^*S\}. \tag{4.8}$$

It follows that

$$\prod_{H \in \mathcal{H}_S} \{H\} \times \mathcal{J}_{S,H} \subset \mathcal{N}_{\mathrm{reg}}(S) \subset \mathcal{H} \times \mathcal{J}^c,$$

$$\prod_{S \in \mathrm{Emb}^H} \{S\} \times \mathcal{J}_{H,S} \subset \mathcal{N}_{\mathrm{reg}}(H) \subset \mathrm{Emb}(S_0 : M) \times \mathcal{J}^c,$$

and all the inclusions are dense.

5. Floer homology of submanifolds

Let $H \in \mathcal{H}_0$, $S \in \mathrm{Emb}^H$ and $J \in \mathcal{J}_{H,S}$. The gradient trajectories of $\mathcal{A}_H\big|_{\Omega(S)}$ on $\Omega(S)$ with respect to the metric $\langle\langle\ ,\ \rangle\rangle_J$ defined as in (3.1) are solutions of the following perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\big(\frac{\partial u}{\partial t} - X_H(u)\big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S. \end{cases} \tag{5.1}$$

We denote by $\mathcal{M}_J(H, S)$ the set of bounded solutions of (5.1), i.e., those with

$$\inf_{\tau \in \mathbb{R}} \mathcal{A}_H(u(\tau)) > -\infty, \quad \sup_{\tau \in \mathbb{R}} \mathcal{A}_H(u(\tau)) < \infty. \tag{5.2}$$

By the monotonically decreasing property of $\mathcal{A}_H$ along the trajectory, we also have

$$\inf_{\tau\in\mathbb{R}} \mathcal{A}_H(u(\tau)) = \lim_{\tau\to\infty} \mathcal{A}_H(u(\tau)),$$
$$\sup_{\tau\in\mathbb{R}} \mathcal{A}_H(u(\tau)) = \lim_{\tau\to\infty} \mathcal{A}_H(u(\tau)).$$

Note that the stationary points for the flow (5.1) are solutions of the Hamilton's equation

$$\begin{cases} \dot z = X_H(z), \\ z(0) \in o_M, \;\; z(1) \in N^*S, \end{cases} \tag{5.3}$$

and each of them can be written as

$$z(t) = \phi_H^t(\phi_H^{-1}(p)) = z_H^p(t)$$

for some $p \in o_M \cap N^*S$ and vice versa. We define

$$\begin{aligned} CF(H, S : M) &= \{z : [0,1] \to T^*M \mid z \text{ solves } (5.3) \\ &= \{z_H^p \mid p \in o_M \cap N^*S\}. \end{aligned}$$

Because of the choice of $H \in \mathcal{H}_0$, $S \in \mathrm{Emb}^H$, there are only finitely many elements in $CF(H, S : M)$. The following two theorems, will be proven in the end of this section. We would like to emphasize that unlikely from the case of periodic orbit problems neither of the (canonical) grading nor the coherent orientations exists in the general relative Floer theory as in [18] or [38]. (See [19] for the non-canonical grading assigned for the geometric version of the Floer homology of Lagrangian intersections.) The existence of these canonical grading and the coherent orientations in our relative Floer theory soly depends on the special circumstance that we are looking at the Lagrangian submanifolds o_M and N^*S *in the cotangent bundles* T^*M. Some more detailed discussions on the coherent orientation in relation to the Poincaré duality will be given in [43], and the complete treatment of the coherent orientation question will be carried out elsewhere in the more general context of Fukaya's A^∞-structure. In this paper, since this orientation question is not the main issue, we will be content to give the complete proof of the orientability of the Floer cells $\mathcal{M}_J(H, S : z^\alpha, z^\beta)$ and to refer to [22], [23] for the details of providing the coherent orientations that are compatible to the gluing procedure. Those who feel uncomfortable about the coherent orientation can safely take $\mathbb{Z}_2$-coefficients for the Floer homology at least in this paper, but we believe that it will be important to define the Floer homology with arbitrary coefficients for the more elaborate applications in the future.

Theorem 5.1 [Canonical grading]. *For each solution z of (5.3), there exists a canonically assigned Maslov index that has the values in $\frac{1}{2}\mathbb{Z}$. We denote this map by*

$$\mu_S : CF(H,S:M) \to \tfrac{1}{2}\mathbb{Z}.$$

Furthermore, μ_S has the following properties:

(1) $\mu_S + \frac{1}{2}\dim S \in \mathbb{Z}$ *and for each solution u of* (5.1) *with* $u(-\infty) = z^\alpha$, $u(+\infty) = z^\beta$, *we have the Fredholm index of u given by*

$$\text{Index} u = \mu_S(z^\alpha) - \mu_S(z^\beta). \tag{5.4}$$

(2) *Consider the time-independent Hamiltonian $F = f \circ \pi$, $f \in C^\infty(M)$ which is defined as described after Theorem 5.5 below. Let $p \in \text{Graph}\, df \cap N^*S$ and so $x = \pi(p) \in Crit(f\big|_S)$. Denote by $z_x(t) = (x, tdf(x))$ which is the Hamiltonian path of F with $z_x(0) \in o_M$, $z_x(1) \in N^*S$. Then we have*

$$\mu_S(z_x) = \mu_f^S(x) - \tfrac{1}{2}\dim S \tag{5.5}$$

where μ_f^S is the Morse index of $f\big|_S$ at x on S.

Theorem 5.2 [Coherent orientation]. (1) *Let (H,S,J) be generic in the isotopy class $[H,S,J]$. For each $z^\alpha, z^\beta \in CF(H,S:M)$, there exists an orientation of $\mathcal{M}_J(z^\alpha,z^\beta)$, i.e., the determinant bundle*

$$\mathbf{Det} \to \mathcal{M}_J(z^\alpha, z^\beta)$$

whose fiber at $u \in \mathcal{M}_J(z^\alpha,z^\beta)$ is the one-dimensional real vector space

$$\det(D\overline{\partial}_{J,H}(u)) := \Lambda^{\max}(\text{Ker } D\overline{\partial}_{J,H}(u)) \otimes \Lambda^{\max}(\text{Coker } D\overline{\partial}_{J,H}(u))$$

is trivial. The same is true for the parametrized version of the Floer cells $\mathcal{M}(\overline{H},\overline{S},\overline{J})$ for the generic paths $(\overline{H},\overline{S},\overline{J})$ as in Section 4.

(2) *Furthermore there exist a coherent orientation (in the sense of [22], [23]) on the set of all $\mathcal{M}_J(H,S)$'s and $\mathcal{M}(\overline{H},\overline{S},\overline{J})$ over (H,S,J)*

and the paths $(\overline{H}, \overline{S}, \overline{J})$ *in each isotopy class. We denote the set of such coherent orientations by* $\mathrm{Or}([H,S,J] : M) = \mathrm{Or}([S] : M)$.

For the moment, we postpone the proofs of these theorems until the end of this section and proceed. We define the grading on $CF(H, S : M)$ by

$$k = \mu_S(z) + \tfrac{1}{2}\dim S$$

and form a $\mathbb{Z}$-graded free abelian group (i.e., $\mathbb{Z}$-module) $CF_*(H, S : M)$. In fact, we can also consider free G-module for any abelian group G.

It is also possible and maybe more natural to give the grading to $CF(H, S : M)$ by the Maslov index itself not by the above formula, allowing the shift by the half integer $\frac{1}{2}\dim S$, when we consider the assignment $(H, S : M) \mapsto CF(H, S : M)$ as a "functor" in a categorical approach. Compare this with the grading provided in the non-relative theory in [54], [7], [52] and [47]. But in this paper, we prefer to use the above integer grading which will coincide with the grading in the singular homology under the isomorphism in Theorem 5.5 below and which makes it easier to keep track of the grading under the pants product in [43].

We fix a coherent orientation $\sigma \in \mathrm{Or}([S] : M)$. Now for each $z^\alpha, z^\beta \in CF(H, S : M)$ with $\mu(z^\alpha) - \mu(z^\beta) = 1$ each element $u \in \mathcal{M}_J(z^\alpha, z^\beta)$ defines its flow orientation $[u_\tau]$. We compare this flow orientation $[u_\tau]$ of the flow with the orientation $\sigma(u)$ induced from the coherent orientation defined in Theorem 5.2, we define the sign $\tau(u) \in \{1, -1\}$ by

$$\sigma(u) = \tau(u)[u_\tau].$$

We define for such z^α and z^β

$$n^\sigma_{(H,J)}(z^\alpha, z^\beta) := \sum_{u \in \mathcal{M}_J(z^\alpha, z^\beta)} \tau(u) \tag{5.6}$$

and a homomorphism $\partial_{(H,J)} : CF_*(H, S : M) \to CF_*(H, S : M)$ by

$$\partial^\sigma_{(H,J)}(z^\alpha) = \sum_\beta n_{(H,J)}(z^\alpha, z^\beta) z^\beta. \tag{5.7}$$

By definition, $\partial^\sigma_{(H,J)}$ has degree -1 with respect to the grading given by Theorem 5.1. By the standard compactness and cobordism argument (see [21], [35] or [38]), we can prove that $\partial^\sigma_{(H,J)}$ satisfies

$$\partial^\sigma_{(H,J)} \circ \partial^\sigma_{(H,J)} = 0,$$

and so we are given a graded complex $(CF_*(H, S : M), \partial^\sigma_{(H,J)})$.

In the present paper, we will mainly concern $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients, and unless otherwise stated, we will always take $\mathbb{Z}$ as the coefficient.

Definition 5.3. We define, for each regular parameter (H, S, J),

$$HF^\sigma_*(H, S, J : M) = \text{Ker } \partial^\sigma_{(H,J)} / \text{Im } \partial^\sigma_{(H,J)}$$

and call it the *Floer homology* of (H, S, J) on M (with respect to the coherent orientation σ).

The following theorem can be proven again by the standard compactness argument.

Theorem 5.4. *For two regular parameters $(H^\alpha, S^\alpha, J^\alpha)$ and $(H^\beta, S^\beta, J^\beta)$ isotopic to each other, there is the canonical isomorphism*

$$h^\sigma_{\alpha\beta} : HF^\sigma_*(H^\alpha, S^\alpha, J^\alpha : M) \to HF^\sigma_*(H^\beta, S^\beta, J^\beta : M)$$

that preserves the grading.

The proof of this theorem follows ideas from [21] and [38]. But we would like to recall how the construction of the isomorphism goes because when we study the filtration, it will be essential to understand the "best" way of choosing the chain homomorphism between $CF_*(H^\alpha, S^\alpha)$ and $CF_*(H^\beta, S^\beta)$ that induces the isomorphism in Theorem 5.4.

We fix a monotone function $\rho : \mathbb{R} \to [0, 1]$ such that

$$\begin{aligned} \rho(\tau) &= 0 \qquad \text{if } \tau < -1, \\ &= 1 \qquad \text{if } \tau > 1, \end{aligned}$$

and define $\rho_K(\tau) = \rho(\tau/K)$ for $K > 0$. We choose a path from $[0, 1]$ to the parameter space

$$\begin{aligned} \{(H^s, S^s, J^s) \mid s \in [0,1], (H^0, S^0, J^0) =& (H^\alpha, S^\alpha, J^\alpha), \\ (H^1, S^1, J^1) =& (H^\beta, S^\beta, J^\beta)\} \end{aligned}$$

such that if we denote

$$(\overline{H}, \overline{S}, \overline{J}) = \{(H^{\rho_K(\tau)}, S^{\rho_K(\tau)}, J^{\rho_K(\tau)})\}_{-\infty \le \tau \le \infty},$$

then all the solutions of the equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + \overline{J}\big(\frac{\partial u}{\partial t} - X_{\overline{H}}(u)\big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*\overline{S} \end{cases} \tag{5.8}$$

become regular for sufficiently large $K > 0$. Again the space of $(\overline{H}, \overline{S}, \overline{J}, K)$ will be dense among the set of paths connecting $(H^\alpha, S^\alpha, J^\alpha)$, $(H^\beta, S^\beta, J^\beta)$ and $K \in \mathbb{R}$.

For each given $z^\alpha \in CF_*(H^\alpha, S^\alpha)$ and $z^\beta \in CF_*(H^\beta, S^\beta)$, we define

$$\mathcal{M}_K(z^\alpha, z^\beta) = \{u : \Theta \to T^*M \mid u \text{ solves (5.8) and } \lim_{\tau\to-\infty} u(\tau) = z^\alpha, \lim_{\tau\to+\infty} u(\tau) = z^\beta\}.$$

Using the orientations provided by Theorem 5.2, we define an integer similarly as in (5.6)

$$n^\sigma_{\alpha\beta}(z^\alpha, z^\beta) := \#(\mathcal{M}_K(z^\alpha, z^\beta)) \quad \text{for } \mu(z^\alpha) - \mu(z^\beta) = 0,$$

and the chain map $h_{\alpha\beta} : CF_*(H^\alpha, S^\alpha) \to CF_*(H^\beta, S^\beta)$ by

$$h^\sigma_{\alpha\beta}(z^\alpha) = \sum n^\sigma_{\alpha\beta}(z^\alpha, z^\beta) z^\beta. \tag{5.9}$$

We would like to emphasize that from the definition, *only those pairs* (z^α, z^β) *for which the equation (5.8) has a solution with the given asymptotic condition*

$$\lim_{\tau\to-\infty} u(\tau) = z^\alpha, \quad \lim_{\tau\to+\infty} u(\tau) = z^\beta$$

give nontrivial contribution in (5.9). This fact is the one which enables us later to estimate the change of the filtration under the various homomorphisms between the Floer homology for different parameters.

Finally, we also have the following theorem whose proof will be a modification, taking the canonical coherent orientation into account, to that in [46] which in turn follows Floer's idea in [20].

Theorem 5.5. *Let (H, S, J) be regular and fix the coherent orientation $\sigma \in \mathrm{Or}([S] : M)$ provided as in Section 4 [22]. Then there exists an isomorphism*

$$F_{(H,S,J)} : H_*(S, \mathbb{Z}) \to HF^\sigma_*(H, S, J : M)$$

that preserves grading, where $H_*(S, \mathbb{Z})$ is the singular homology of M. In particular, $HF_*(H, S, J : M) \neq \{0\}$. We will call this coherent orientation the canonical coherent orientation.

We briefly outline the idea of the proof in [20] and [46], incorporating the coherent orientation, to explain how the filtration on the

Floer homology $HF^*(H,S,J:M)$ is affected by the homomorphism $F_{(H,S,J)} : H_*(S,\mathbb{Z}) \to HF_*(H,S,J:M)$. We first choose a tubular neighborhood U of S which we identify with the normal bundle $\pi_0 : NS \to S$. Then we choose a smooth function f_0 on S that is of Morse-Smale type and consider the function $f_0 \circ \pi_0$ on U. We extend this to M by a cut-off function and denote the extension by f. We now define the (time-independent) Hamiltonian

$$H_0 = f \circ \pi : T^*M \to \mathbb{R}.$$

Then it is easy to prove, following the idea in [20] and [46], the solutions of the equation

$$\begin{cases} \dot{z} = X_{H_0}(z), \\ z(0) \in o_M, \quad z(1) \in N^*S \end{cases}$$

have one-to-one correspondence with the critical points of $f\big|_S = f_0$, the restriction of f to S. Furthermore, following the idea of [20] one can prove that any element $u \in \mathcal{M}_{J_g}(H_0,S)$ is t-independent provided $|f|_{C^2}$ is sufficiently small. By the equation (1.6) and the choice of H_0, we see that any such u has the form

$$u(\tau,t) = \chi(\tau)$$

for some gradient trajectory χ of $f|_S = f_0$. Once this is proven, the assignment

$$\chi \in \mathcal{M}_g(f|_S) \to u(\tau,t) := \chi(\tau)$$

provides a natural diffeomorphism between the Morse complex $\mathcal{M}_g(f\big|_S)$ and $\mathcal{M}_{J_g}(H_0,S)$. Furthermore under this natural diffeomorphism, the canonical coherent orientation σ given in Theorem 5.5 induces a coherent orientation on $\mathcal{M}_g(f\big|_S)$ that coincides with the standard orientation as in Section 7 [37] which is provided by giving orientations to the unstable manifolds of the Morse complex. Then combining this with Theorem 7.4 [37] (from which one can easily derive that the Morse homology with this coherent orientation on the Morse complex is isomorphic to the singular homology), we obtain that the above diffeomorphism induces a natural isomorphism between $H_*(S,\mathbb{Z})$ and $HF_*^\sigma(H_0,S,J_g:M)$. For the general (H,S,J), we apply Theorem 5.2 to $HF_*^\sigma(H_0,S,J_g:M)$ and $HF_*^\sigma(H,S,J:M)$. We would like to note that by making $|f|_{C^2}$ as small as we want, the width of the action spectrum $\mathrm{Spec}(H_0,S)$ can be made arbitrarily close to zero.

One immediate corollary of Theorem 5.5 is

Corollary 5.6. *For any* $(H, S) \in \mathcal{N}_{\text{reg}}$*, the following holds*

$$\#(N^*S \cap \phi_H(o_M)) \geq \text{rank } H_*(S, \mathbb{Z}),$$

provided $N^*S \pitchfork \phi_H(o_M)$.

This corollary can be interpreted as that the cycle $\Omega(S)$ is linked to the fundamental cycle $\Omega(M)$. It is also a consequence of the existence of generating function quadratic at infinity.

Example 5.7. Let σ be the canonical coherent orientation as above.

(1) When $S = M$, we have

$$HF_*(H, S, J : M) = HF_*(H, M, J : M) \simeq H_*(M, \mathbb{Z}).$$

(2) When $S = \{pt\}$, we have

$$HF_*^\sigma(H, \{q\}, J : M) \simeq \mathbb{Z}.$$

(3) When $S \simeq S^1$, we have

$$HF_*^\sigma(H, S, J : M) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

Now, we go back to the questions of grading and orientations.

5.1. Canonical grading.

Let $S \subset M$ be a given compact submanifold. We will assign a canonically defined *half-integer*, which we call the *Maslov index* of the Hamiltonian path $z : [0, 1] \to T^*M$, which is a solution of

$$\begin{cases} \dot{z} = X_H(z) \\ z(0) \in o_M, \quad z(1) \in N^*S. \end{cases} \tag{5.10}$$

The definition of the Maslov index of general pairs of Lagrangian paths in $\mathbb{R}^{2n}$ ($\simeq \mathbb{C}^n$) has been given by [10] or [49], which we will use to define the Maslov index for solutions of (5.10). We will mostly follow the exposition given in [49], [50] except slight differences of the conventions and the notation used. We will define the Maslov index of z by trivializing the vector bundle z^*TX over $[0,1]$.

One could attempt to define the Maslov index of the Hamiltonian paths in general (P,w) with general boundary conditions $L_0, L_1 \subset P$, i.e., of the solutions

$$\begin{cases} \dot{z} = X_H(z), \\ z(0) \in L_0,\ z(1) \in L_1. \end{cases} \tag{5.11}$$

However in this generality, there will be *no canonical* definition of the index since the definition in general depends on the choice of trivialization.

The crucial observation of ours is that on $X = T^*M$, there is a certain canonical class of symplectic trivialization

$$\Phi : z^*TX \to [0,1] \times \mathbb{C}^n$$

due to the fact that we are given a *fixed* Riemannian metric g on M and so the tangent bundle TX has the canonical splitting as the sum of Lagrangian subbundle

$$TX = H \oplus V,$$

where V is the vertical tangent bundle the fiber V_p of which is canonically isomorphic to $T^*_{\pi(p)}M$, and H is the horizontal subbundle with respect to the Levi-Civita connection of g the fiber H_p of which is isomorphic to $T_{\pi(p)}M$ under the map $T\pi : TX \to TM$.

We now consider the class of symplectic trivializations $\Phi : z^*TX \to [0,1] \times \mathbb{R}^n \oplus (\mathbb{R}^n)^* \cong [0,1] \times \mathbb{C}^n$ that satisfies

$$\Phi(H_{z(t)}) \equiv \mathbb{R}^n, \quad \Phi(V_{z(t)}) \equiv (\mathbb{R}^n)^* \cong i\mathbb{R}^n \tag{5.12}$$

for all $t \in [0,1]$; we denote the class by $\mathcal{T}$. Such a trivialization always exists because $[0,1]$ is contractible. For example, such a Φ can be obtained by the parallel transport along the paths which are the composition of two linear paths

$$(0,0) \to (\tau,0) \to (\tau,t).$$

Here the parallel transport is with respect to the natural connection on u^*TX induced by the Levi-Civita connection with respect to the metric g on M. The transition map between two such trivializations Φ and Ψ in $\mathcal{T}$ is given by the form

$$\Psi \circ \Phi^{-1}(t,v) = (t, A_{\Psi\Phi}(t)v),\ A_{\Psi\Phi}(t) = Q(t) \oplus (Q^*(t))^{-1}, \tag{5.13}$$

where $Q(t) \in \mathrm{GL}(n,\mathbb{R})$ and $Q^*(t)$ is the conjugate to $Q(t)$. In other words, we have reduced the structure group of z^*TX from $\mathrm{Sp}(2n,\mathbb{R})$ to a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ that is isomorphic to $\mathrm{GL}(n,\mathbb{R})$. Now note that each solution z of (5.10) has the form

$$z(t) = \phi_H^t(\phi_H^{-1}(p)), \quad p \in \phi_H(o_M) \cap N^*S,$$

where ϕ_H^t is the Hamiltonian flow of H. For given such z, we choose a trivialization $\Phi \in \mathcal{T}$. In this trivialization, we will have

$$\Phi(T_{z(0)}o_M) = \mathbb{R}^n, \;\; \Phi(T_{z(1)}N^*S) = U_\Phi \oplus U_\Phi^\perp, \tag{5.14}$$

where $U_\Phi \subset \mathbb{R}^n$ is a $k = \dim S$ dimensional subspace and $(U_\Phi)^\perp \subset (\mathbb{R}^n)^*$ is the annihilator of U_Φ. We denote

$$V^\Phi := \Phi(T_{z(1)}N^*S) = U_\Phi \oplus U_\Phi^\perp \tag{5.15}$$

and define the symplectic path $B_\Phi : [0,1] \to \mathrm{Sp}(2n)$ by

$$B_\Phi(t) := \Phi \circ T\phi_H^t \circ \Phi^{-1} : \mathbb{C}^n \cong \{0\} \times \mathbb{C}^n \to \{t\} \times \mathbb{C}^n \cong \mathbb{C}^n. \tag{5.16}$$

Following the definition of [49], we now consider the Maslov index

$$\mu(\mathrm{Gr}(B_\Phi), \mathbb{R}^n \oplus V^\Phi),$$

which becomes the same as $\mu(B_\Phi(\mathbb{R}^n), V^\Phi)$.

The following lemma is the crucial lemma that enables us to define the canonical grading in this special circumstances of (5.10), which will not exist in the general context of (5.11).

Lemma 5.8. *If Φ, $\Psi \in \mathcal{T}$, then*

$$\mu(B_\Phi(\mathbb{R}^n), V^\Phi) = \mu(B_\Psi(\mathbb{R}^n), V^\Psi).$$

Proof . From (5.15) and (5.16), it follows that

$$V^\Psi = A_{\Psi\Phi}(1) \cdot V^\Phi,$$
$$B_\Psi(t) = A_{\Psi\Phi}(t) B_\Phi(t) A_{\Psi\Phi}(0)^{-1},$$

so that

$$\mu(B_\Psi(t) \cdot \mathbb{R}^n, V^\Psi) = \mu(A_{\Psi\Phi}(t) B_\Phi(t) A_{\Psi\Phi}(0)^{-1}\mathbb{R}^n, A_{\Psi\Phi}(1) \cdot V^\Phi).$$

Using the fact that $A_{\Psi\Phi}(t)$ has the block diagonal form as in (5.13), we have

$$A_{\Psi\Phi}(0)^{-1} \cdot \mathbb{R}^n \equiv \mathbb{R}^n.$$

Applying this and the naturality axiom, Theorem 3.1 [49], yields

$$\mu(B_\Psi(A) \cdot \mathbb{R}^n, V^\Psi) = \mu(\mathbb{R}^n, B_\Phi(t)^{-1} A_{\Psi\Phi}(t)^{-1} A_{\Psi\Phi}(1) \cdot V^\Phi). \tag{5.17}$$

Now, we consider the homotopy $\{B_s\}_{0 \le s \le 1}$ defined by

$$B_s(t) := B_\Phi(t)^{-1} A_{\Psi\Phi}(st)^{-1} A_{\Psi\Phi}(s), \quad t \in [0,1]$$

and the Lagrangian path $\Lambda_s(t) := B_s(t) \cdot V^\Phi$. This homotopy has the property that for all s

$$\begin{aligned}
\Lambda_s(0) &= B_\Phi(0)^{-1} A_{\Psi\Phi}(s) \cdot V^\Phi = A_{\Psi\Phi}(s) \cdot V^\Phi \\
&= (Q(s) \oplus Q^*(s)^{-1})(U_\Phi \oplus U_\Phi^\perp) \\
&= (Q(s)U_\Phi) \oplus (Q(s)U_\Phi)^\perp \in \Sigma_k(\mathbb{R}^n), \\
\Lambda_s(1) &= B_\Phi(1)^{-1} \cdot V^\Phi \in \Sigma_\ell(\mathbb{R}^n),
\end{aligned}$$

where $k = \dim S$, $\ell = \dim T\phi_H^1(T_{z(0)} o_M) \cap T_{z(1)}(N^* S)$ and

$$\Sigma_k(\mathbb{R}^n) := \{V \in \Lambda(n) \mid \dim V \cap \mathbb{R}^n = k\}.$$

In other words, the homotopy Λ_s is *a stratum homotopy with respect to* $\mathbb{R}^n$ in the sense of [49]. By Theorem 2.4 [49], we conclude $\mu(\mathbb{R}^n, \Lambda_0) = \mu(\mathbb{R}^n, \Lambda_1)$. However we have

$$\begin{aligned}
\Lambda_0(t) &= B_\Phi(t)^{-1} \cdot V^\Phi, \\
\Lambda_1(t) &= B_\Phi(t)^{-1} A_{\Psi\Phi}(t)^{-1} A_{\Psi\Phi}(1) \cdot V^\Phi,
\end{aligned}$$

and hence

$$\mu(\mathbb{R}^n, B_\Phi(t)^{-1} \cdot V^\Phi) = \mu(\mathbb{R}^n, B_\Phi(t)^{-1} A_{\Psi\Phi}(t)^{-1} A_{\Psi\Phi}(1) \cdot V^\Phi).$$

Combining this with (5.17), we have proven

$$\mu(B_{\Psi(t)} \cdot \mathbb{R}^n, V^\Psi) = \mu(\mathbb{R}^n, B_{\Phi(t)^{-1}} \cdot V^\Phi).$$

By applying the naturality axiom Theorem 3.1 [49] to the right-hand side again, we have finished the proof. q.e.d.

Now, we are ready to define the canonical Maslov index of z.

Definition 5.9. The *Maslov index* of z of a solution of (5.10) is denoted as $\mu_S(z)$ and given by

$$\mu_S(z) = -\mu(\mathrm{Gr}(B_\Phi), \mathbb{R}^n \oplus V^\Phi) = -\mu(B_\Phi \cdot \mathbb{R}^n, V^\Phi)$$

for a trivialization (and so for any trivialization) Φ in $\mathcal{T}$.

The sign is so chosen that we have (5.4) and (5.5) in Theorem 5.1 not with the opposite signs.

Proof of Theorem 5.1 (1). The statement $\mu_S + 1/2 \dim S \in \mathbb{Z}$ is an immediate consequence of Theorem 2.4 [49], and so we will just prove the second statement. Let $u : \mathbb{R} \times [0,1] \to T^*M$ be a solution of

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\big(\frac{\partial u}{\partial t} - X_H(u)\big) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S. \end{cases} \tag{5.18}$$

We denote

$$\overline{\partial}_{J,H}(u) := \frac{\partial u}{\partial \tau} + J\big(\frac{\partial u}{\partial t} - X_H(u)\big).$$

Under the assumption

$$\phi_H^1(o_M) \pitchfork N^*S$$

the linearization operator $E_u : T_u\mathcal{F}^{1,2} \to \mathcal{L}_u^2$ becomes a Fredholm operator. We recall

$$\begin{aligned} \mathcal{W}_u^{1,2} &:= T_u\mathcal{F}^{1,2} \\ &= \{\xi \in W^{1,2}(u^*TX) \mid \xi(\tau,0) \in T_{u(\tau,0)}\mathcal{O}, \xi(\tau,1) \in T_{u(\tau,1)}(N^*S)\}, \\ \mathcal{L}_u^2 &= L^2(u^*TX), \end{aligned}$$

and $E_u := D\overline{\partial}_{J,H}(u)$, the covariant linearization of $\overline{\partial}_{J,H}$ with respect to the canonical connection induced from the Levi-Civita connection on $X = T^*M$. Again we trivialize the bundle u^*TX so that

$$\Phi(H_{u(\tau,t)}) \equiv \mathbb{R}^n, \quad \Phi(V_{u(\tau,t)}) \equiv (\mathbb{R}^n)^* \tag{5.19}$$

for all (τ, t), which is again possible because $\mathbb{R} \times [0,1]$ is contractible. We denote by $\mathrm{Index}_\Phi(u)$ the Fredholm index of the push-forward operator $L_u^\Phi := \Phi_* E_u : W_{\Lambda^\Phi}^{1,2} \to L$ where

$$\begin{aligned} W_{\Lambda^\Phi}^{1,2} :=& \{\zeta \in W^{1,2}(\mathbb{R} \times [0,1], \mathbb{C}^n) \mid \zeta(\tau,0) \in \mathbb{R}^n, \zeta(\tau,1) \in \Lambda^\Phi(\tau)\}, \\ L^2 =& L^2(\mathbb{R} \times [0,1], \mathbb{C}^n). \end{aligned}$$

To compare $\text{Index}_\Phi(u)$ and $\mu_S(u(\pm\infty))$ defined in Definition 5.9, we require that the trivialization

$$\Phi : u^*TX \to \mathbb{R} \times [0,1] \times \mathbb{C}^n$$

extends the trivializations at $\tau = \pm\infty$ fixed in advance

$$\Phi_\pm : z_\pm^* TX \to [0,1] \times \mathbb{C}^n, \;\; z_\pm = u(\pm\infty),$$

which are in the class $\mathcal{T}$. Such an extension is always possible because one can prove that the class $\mathcal{T}$ is connected in an obvious sense. Next, we consider the push-forward operator $\Phi_* E_u : W^{1,2}_{\Lambda^\Phi} \to L^2$. By a straightforward computation, one can prove that this operator becomes an operator of the Cauchy-Riemann type

$$\begin{cases} \overline{\partial}_{J,T,\Lambda_1}\zeta := \frac{\partial \zeta}{\partial t} + J\frac{\partial \zeta}{\partial t} + T\zeta, \\ \zeta(\tau,0) \in \mathbb{R}^n, \;\; \zeta(\tau,1) \in \Lambda_1^\Phi(\tau), \end{cases}$$

where J and T satisfy the following properties (Compare with the conditions (CR-1,2,3) in Section 7 [50].):

(1) *The almost complex structures* $J : \mathbb{R} \times [0,1] \to \text{End}\ (\mathbb{R}^{2n})$ *satisfy*

$$\lim_{\tau\to\pm\infty} \sup_{0\le t\le 1} \|J(\tau,t) - J(\pm\infty,t)\| = 0. \tag{5.20}$$

(2) *The function* $T : \mathbb{R} \times [0,1] \to \text{End}\ (\mathbb{R}^{2n})$ *satisfies*

$$\lim_{\tau\to\pm\infty} \sup_{0\le t\le 1} \|T(\tau,t) - T(\pm\infty,t)\| = 0. \tag{5.21}$$

We denote the asymptotic operators $T(\pm,\cdot)$ *by* $T_\pm$.

(3) *Let* $DX_H(z)$ *be the covariant linearization of* X_H *along the Hamiltonian path* z. *Then*

$$(\Phi_\pm)_* DX_H(z_\pm) = J(\pm\infty,t)\cdot T(\pm\infty,t).$$

(4) *Let* $\Psi_{T_\pm} : [0,1] \to Sp(2n)$ *be defined by the equations*

$$\frac{\partial \Psi_{T_\pm}}{\partial t} - J(\pm\infty,t)T(\pm\infty,t)\Psi_{T_\pm} = 0,$$
$$\Psi_{T_\pm}(0) = \mathbb{I}.$$

Then the Lagrangian subspaces $\Psi_{T_\pm}(\mathbb{R}^n)$ are transverse to $\Lambda^{\Phi_\pm}$ respectively.

We denote, for each (J, T, Λ), its asymptotic limits by $(J_\pm, T_\pm, \Lambda_\pm)$ and the asymptotic operators of $\overline{\partial}_{J,T,\Lambda}$ by

$$\begin{aligned} A_\pm &= J_\pm \frac{\partial}{\partial t} + T_\pm \quad \text{on} \\ W^{1,2}_\pm &= \{\xi \in W^{1,2}([0,1], \mathbb{C}^n) \mid \xi(0) \in \mathbb{R}^n,\ \xi(1) \in \Lambda_\pm\} \quad \text{respectively.} \end{aligned}$$

Now, we quote a theorem from [50] that applies to the class of operators considered above.

Lemma 5.10 [Theorem 7.42 [50]]. *The Fredholm operator $\overline{\partial}_{J,T,\Lambda^\Phi} : W^{1,2}_{\Lambda^\Phi} \to L^2$ has the index given by*

$$\begin{aligned} \text{Index}\, \overline{\partial}_{J,T,\Lambda^\Phi} &= -\mu(\ Gr(\Psi^-), \mathbb{R}^n \oplus \Lambda^{\Phi_-}) + \mu(Gr(\Psi^+), \mathbb{R}^n \oplus \Lambda^{\Phi_+}) \\ &\quad + \mu(\Delta, \mathbb{R}^n \oplus \Lambda^\Phi) \\ &= -\mu(\Psi^- \cdot \mathbb{R}^n, \Lambda^{\Phi_-}) + \mu(\Psi^+ \cdot \mathbb{R}^n, \Lambda^{\Phi_+}) + \mu(\mathbb{R}^n, \Lambda^\Phi), \end{aligned} \tag{5.22}$$

where Δ is the diagonal in $\mathbb{C}^n \oplus \mathbb{C}^n$.

Remark 5.11. We would like to note that in [50] the authors considered operators of the type, in our notation,

$$\frac{\partial}{\partial \tau} - J \frac{\partial}{\partial t} + S.$$

Incorporation of these differences change the signs of the terms in the formula from Theorem 7.42 [50].

We now note that in (5.22), the first two terms are exactly,

$$\mu_S(u(-\infty)) - \mu_S(u(+\infty))$$

and therefore to prove Theorem 5.1(1), we have only to prove

$$\mu(\mathbb{R}^n, \Lambda^\Phi) = 0.$$

However, from the definition $\Lambda^\Phi(\tau) = \Phi(T_{u(\tau,1)}(N^*S))$ and the way we choose the trivialization Φ, we have

$$\Lambda^\Phi(\tau) = U_\Phi(\tau) \oplus (U_\Phi(\tau))^\perp,$$

where $U_\Phi(\tau) \subset \mathbb{R}^n$ is a subspace of dimension $k = \dim S$. In other words, $\Lambda_1^\Phi(\tau)$ lies in the same fixed stratum $\Sigma_k(\mathbb{R}^n)$ for all τ. Now the zero axiom from [49] immediately implies

$$\mu(\mathbb{R}^n, \Lambda^\Phi) = 0,$$

which finishes the proof. q.e.d.

Proof of Theorem 5.1(2). Recall first that the Hamiltonian flow of the Hamiltonian $F = f \circ \pi$ is just the vertical linear translation given by

$$\begin{aligned}(q,p) \mapsto& (q, p + tdf(q)), \quad \text{i.e.,} \\ \phi_F^t(q,p) =& (q, p + tdf(q)),\end{aligned} \tag{5.23}$$

where $f : M \to \mathbb{R}$ is a smooth function. When we choose $f(x,\theta) = f_S(x)$ in a tubular neighborhood identified with the normal bundle of S in M, if $\phi_F^1(q,p) \in N^*S$, then we have

$$\begin{aligned}\theta =& 0 \ \ (\text{i.e., } q = (x,0) \in S), \\ df_S(x) =& 0 \ \ (\text{i.e., } x \text{ is a critical point of } f_S).\end{aligned}$$

The corresponding Hamiltonian path is given by

$$z_x(t) = (x, 0, 0, 0)$$

in this splitting. By choosing the canonical coordinates around x, i.e., on $T^*M\big|_U$ where U is a neighborhood of x in M, we may assume that $M = \mathbb{R}^n$, $S = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ and (x,θ) is the coordinates in terms of the splitting $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$.

We denote the corresponding conjugate coordinates by (p_x, p_θ). In terms of this coordinates, the map (5.23) can be written as

$$\phi_F^t(x, \theta, p_x, p_\theta) = (x, \theta, p_x + t\frac{\partial f_s}{\partial x}, p_\theta). \tag{5.24}$$

Therefore, we can write

$$T\phi_F^t(x,\theta,p_x,p_\theta) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ td^2 f_S(x) & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \tag{5.25}$$

in this coordinates. On the other hand, we have

$$V^\Phi = (\mathbb{R}^k \oplus \{0\}) \oplus (\{0\} \oplus (\mathbb{R}^{n-k})^*) \tag{5.26}$$

in $\mathbb{R}^n \oplus (\mathbb{R}^n)^* = \mathbb{C}^n$. One can easily check that since we assume $\phi_F^1(o_M) \pitchfork N^*S$, i.e., that $d^2 f_S(x)$ is nondegenerate for each $(x,\theta) \in \phi_F^1(o_M) \cap N^*S$, we can conclude that for $\vec{p} = (x,\theta,p_x,p_\theta)$

$$(T\phi_F^t(\vec{p}), \vec{p}) \in \mathbb{R}^n \oplus V^\Phi$$

if and only if

$$t = 0, \ \ \theta = 0 \ \text{ and } x \in \operatorname{Crit} f_S.$$

We now recall the definition from [49] of the Maslov index for a curve $\Lambda : [a,b] \to \Lambda(n)$ and a fixed V:

$$\mu(\Lambda, V) := \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, V, a) + \sum_{a<t<b} \operatorname{sign} \Gamma(\Lambda, V, t) + \tfrac{1}{2} \operatorname{sign} \Gamma(\Lambda, V, b),$$

where $\Gamma(\Lambda, V, t)$ is a quadratic form defined on $\Lambda(t) \cap V$, which is called the *crossing form* in [49]. Therefore, we have for the path $t \mapsto T\phi_F^t \cdot \mathbb{R}^n, 0 \le t \le 1$,

$$\mu(T\phi_F^t \cdot \mathbb{R}^n, V^\Phi) = \tfrac{1}{2} \operatorname{sign} \Gamma(T\phi_F^t \cdot \mathbb{R}^n, V^\Phi, 0), \tag{5.27}$$

and so it remains to compute the signature of the crossing form $\Gamma(T\phi_F^t \cdot \mathbb{R}^n, V^\Phi, 0)$. Using the expressions (5.23), (5.26) and Theorem 1.1 [49], it is straightforward to check

$$\begin{aligned} \operatorname{sign} \Gamma(T\phi_F^t \cdot \mathbb{R}^n, V^\Phi, 0) &= \operatorname{sign} d^2 f_S(x) \\ &= \dim S - 2\mu_{f_S}(x), \end{aligned}$$

where $\mu_{f_S}(x)$ is the index of $d^2 f_S(x)$. Hence we have

$$\mu_S(z_x) = -\mu(T\phi_F^t \cdot \mathbb{R}^n, V^\Phi) = -\tfrac{1}{2} \dim S + \mu_{f_S}(x),$$

which finishes the proof of Theorem 5.1(2). q.e.d.

5.2. Coherent orientations.

In this section, we will give a complete proof of Theorem 5.2 (1) and follow the line of the reasoning given in [22] for the proof of Theorem 5.2 (2). A more general discussion on the orientation question in the context of A^∞-structure will be given elsewhere.

The central problem in the approach of [22] (or in the orientation problem of other moduli spaces in general) is to prove that the determinant bundle

$$\mathbf{Det} \to \mathcal{M}_J(z^\alpha, z^\beta)$$

is trivial for all $(z^\alpha, z^\beta) \subset CF(H, S : M)$. Once this has been done, the rest of the arguments proving the existence of the coherent orientations compatible to the gluing procedure, is well described in Section 4-5 [22], which can be also applied to our case. In fact, our case will be simpler than [22] in the proof by two reasons: First, there occurs no bubbling in our case and secondly we have a canonical class of trivializations of u^*TX that satisfy (5.19).

We would like to take this chance to emphasize that in the general relative Floer theory where the bubbling phenomenon exists, it is *not true* in general that the above determinant bundle is trivial. Even the simpler *moduli space of pseudo-holomorphic discs* may not be orientable in general, which one should compare with the fact that the *moduli space of pseudo-holomorphic spheres* is *always orientable*. The orientation problem in the relative Floer theory is different from other orientation problems from the periodic orbit problem (in symplectic geometry) or from the gauge theory, in that the former is in the realm of *the index theory of the elliptic boundary value problem*, while others do not involve boundary values. Furthermore neither the moduli space $\mathcal{M}_J(z^\alpha, z^\beta)$ is simply connected, nor the space

$$\mathcal{F}(z^\alpha, z^\beta) = \{u \in C^\infty(\mathbb{R} \times [0,1], T^*M) \mid u(\tau, 0) \in o_M,\, u(\tau, 1) \in N^*S, \\ \lim_{\tau \to -\infty} u(\tau) = z^\alpha,\, \lim_{\tau \to \infty} u(\tau) = z^\beta \}$$

nor the space of the corresponding operator family is contractible in general. Therefore there is no a priori reason why the determinant bundle is trivial and so we really have to carefully analyze the family of operators involved in this index theory. For this purpose, let us first study the linearization operator of $\overline{\partial}_{H,J}$ at $u \in \mathcal{M}_J(H, S)$.

For each $u \in \mathcal{M}_J(z^\alpha, z^\beta) = \mathcal{M}_J(H, S : z^\alpha, z^\beta)$, we consider the linearization operator

$$E_u := D\overline{\partial}_{H,J}(u) : \mathcal{W}^{1,2}_u \to \mathcal{L}^2_u,$$

which we will study through the trivializations $\Phi = \Phi_u : u^*TX \to (\mathbb{R} \times [0,1]) \times \mathbb{C}^n$ that satisfy (5.19). As we have mentioned before, the operator $(\Phi_u)_* E_u = \Phi_u \circ E_u \circ \Phi_u^{-1}$ is of the form

$$\overline{\partial}_{J,T,\Lambda^\Phi} := \frac{\partial}{\partial \tau} + J\frac{\partial}{\partial t} + T$$

that satisfies (1)-(4) in Section 5.1, which acts on the space

$$W^{1,2}_{\Lambda^\Phi} = \left\{ \zeta \in W^{1,2}(\mathbb{R} \times [0,1], \mathbb{C}^n) \;\mid\; \zeta(\tau, 0) \in \mathbb{R}^n, \, \zeta(\tau, 1) \in \Lambda^\Phi(\tau) \right\},$$

where $\Lambda_1^\Phi(\tau) = U^\Phi(\tau) \oplus (U^\Phi(\tau))^\perp \subset \mathbb{R}^n \oplus (\mathbb{R}^n)^*$ with $\dim U^\Phi(\tau) = \dim S =: k$. We denote

$$\begin{aligned}
\mathcal{J} &= \{ J : \mathbb{R} \times [0,1] \to \mathrm{End}(\mathbb{R}^{2n}) \mid J^2 = -id, \quad J \text{ compatible to } \omega_0 \\
&\qquad\qquad \text{and satisfying (5.20)} \}, \\
\mathcal{S} &= \{ T : \mathbb{R} \times [0,1] \to \mathrm{End}(\mathbb{R}^{2n}) \mid T \text{ satisfies (5.21) } \}, \\
\mathcal{L} &= \{ \Lambda : \mathbb{R} \to \Lambda(n) \mid \Lambda(\tau) \to \Lambda(\pm\infty) \text{ as } \tau \to \pm\infty \}, \\
\Sigma &= \{ \Lambda \in \mathcal{L} \mid \Lambda(\tau) \in \Sigma_k(\mathbb{R}^n) \}, \\
\Omega &= \{ \Lambda \in \mathcal{L} \mid \Lambda(\tau) = U(\tau) \oplus (U(\tau))^\perp, \, U(\tau) \in \mathrm{Gr}_k(\mathbb{R}^n) \}.
\end{aligned}$$

Note from the definitions that $\Omega \subset \Sigma \subset \mathcal{L}$. For each given asymptotic operators $A_\pm = (J_\pm, T_\pm, \Lambda_\pm)$, satisfying (4) in Section 5.1, we define

$$\begin{aligned}
(\mathcal{J} \times \mathcal{S} \times \mathcal{L})_{A_\pm} = \{ (J, T, \Lambda) \in \mathcal{J} \times \mathcal{S} \times \mathcal{L} : & (J, T, \Lambda) \text{ satisfies (1), (2)} \\
& \text{and (4) in Section 5.1 } \},
\end{aligned}$$

and $(\mathcal{J} \times \mathcal{S} \times \Sigma)_{A_\pm}$ and $(\mathcal{J} \times \mathcal{S} \times \Omega)_{A_\pm}$ similarly. We also define

$$\mathcal{L}_{\Lambda_\pm} = \{ \Lambda \in \mathcal{L} \mid \Lambda(+\infty) = \Lambda_+, \; \Lambda(-\infty) = \Lambda_- \},$$

and similarly for $\Sigma_{\Lambda_\pm}$ or $\Omega_{\Lambda_\pm}$.

For each $u \in \mathcal{M}_J(z^\alpha, z^\beta)$, we trivialize u^*TX by the parallel transport described as before and denote this canonical trivialization by Φ. Then the assignment $u \mapsto (\Phi)_* E_u$ defines a map from $\mathcal{M}_J(z^\alpha, z^\beta)$ to $(\mathcal{J} \times \mathcal{S} \times \Omega)_{A_{\alpha\beta}} \subset (\mathcal{J} \times \mathcal{S} \times \mathcal{L})_{A_{\alpha\beta}}$ where $A_{\alpha\beta}$ is the asymptotic operators naturally induced from the operators $(\Phi)_* E_u$. Note that all $(\Phi)_* E_u$ have the same asymptotic operators $A_{\alpha\beta}$ as long as $u \in \mathcal{M}_J(z^\alpha, z^\beta)$ for fixed z^α, z^β. We denote this "Gauss" map by

$$G : \mathcal{M}_J(z^\alpha, z^\beta) \to (\mathcal{J} \times \mathcal{S} \times \mathcal{L})_{A_{\alpha\beta}},$$

and then the bundle $\mathbf{Det} \to \mathcal{M}_J(z^\alpha, z^\beta)$ is just the pull-back bundle by the map G of the universal determinant bundle

$$\mathbf{Det} \to (\mathcal{J} \times \mathcal{S} \times \mathcal{L})_{A_{\alpha\beta}}. \tag{5.28}$$

The fiber of this universal bundle is defined as follows: For each

$$(J, T, \Lambda) \in (\mathcal{J} \times \mathcal{S} \times \mathcal{L})_{A_{\alpha\beta}},$$

we form the operator

$$\overline{\partial}_{J,T,\Lambda} = \frac{\partial}{\partial \tau} + J\frac{\partial}{\partial t} + T$$

on the space

$$W^{1,2}_\Lambda = \{\zeta \in W^{1,2}(\mathbb{R} \times [0,1], \mathbb{C}^n) \mid \zeta(\tau, 0) \in \mathbb{R}^n, \zeta(\tau, 1) \in \Lambda\},$$

which becomes a Fredholm operator due to the conditions (1), (2) and (4), and so both Ker $(\overline{\partial}_{J,T,\Lambda})$ and Coker$(\overline{\partial}_{J,T,\Lambda})$ become finite dimensional real vector spaces. Hence we can form the one-dimensional (real) vector space

$$\det(\overline{\partial}_{J,T,\Lambda}) := \wedge^{\max}(\mathrm{Ker}\ \overline{\partial}_{J,T,\Lambda}) \otimes \wedge^{\max}(\mathrm{Coker}\ \overline{\partial}_{J,T,\Lambda}).$$

Since the image of G lies in $(\mathcal{J} \times \mathcal{S} \times \Omega)_{A_{\alpha\beta}}$, to prove the triviality of $\mathbf{Det} \to \mathcal{M}_J(z^\alpha, z^\beta)$, it will be enough to show that the restriction

$$\mathbf{Det} \to (\mathcal{J} \times \mathcal{S} \times \Omega)_{A_{\alpha\beta}} \tag{5.29}$$

of the universal determinant bundle (5.28) is trivial. We start with the following lemma but omit the proof which is an easy consequence of the facts that both the space of compatible almost complex structure and the space of endomorphisms are contractible and that $\Sigma_0(\mathbb{R}^n)$ is contractible.

Lemma 5.12. *The fibration*

$$(\mathcal{J} \times \mathcal{S} \times \mathcal{L})_{A_\pm} \to \mathcal{L}_{\Lambda_\pm}, \quad (J, T, \Lambda) \mapsto \Lambda$$

has (weakly) contractible fibers where the union is over all $A_\pm$ satisfying (1),(2) and (4), and in particular so does the restriction

$$(\mathcal{J} \times \mathcal{S} \times \Omega)_{A_\pm} \to \Omega_{\Lambda_\pm}$$

when $\Lambda_\pm = U_\pm \oplus (U_\pm)^\perp$ with $U_\pm \in Gr_k(\mathbb{R}^n)$.

Noting that when $S = M$, $\Omega = \{$ the constant map, $\mathbb{R}^n\}$ (recall that $\mathrm{Gr}_n(\mathbb{R}^n) = \{\mathbb{R}^n\}$), we have the following immediate consequence of this lemma:

Corollary 5.13. *For each* (z^α, z^β)*, the determinant bundle* $\mathbf{Det} \to \mathcal{M}_J(z^\alpha, z^\beta)$ *is trivial when* $S = M$.

For the general submanifold $S \subset M$, we have to further analyze the inclusion map $\Omega_{\Lambda_\pm} \hookrightarrow \mathcal{L}_{\Lambda_\pm}$. The main proposition then is

Proposition 5.14. *Let* $\Lambda_\pm = U_\pm \oplus (U_\pm)^\perp \in \Sigma_k(\mathbb{R}^n) \subset \Lambda(n)$. *Then the inclusion map* $\Omega_{\Lambda_\pm} \hookrightarrow \mathcal{L}_{\Lambda_\pm}$ *is homotopic to the constant map.*

Once we have proven this, the triviality of the bundle (5.29) will immediately follow because it is the pull-back bundle of the universal bundle (5.28) under the inclusion map

$$(\mathcal{J} \times \mathcal{S} \times \Omega)_{A_\pm} \hookrightarrow (\mathcal{J} \times \mathcal{S} \times \mathcal{L})_{A_\pm}$$

which is homotopic to the constant map. Therefore to finish the proof of Theorem 5.2, it remains to prove Proposition 5.14. In fact, we will prove that the inclusion map

$$\Sigma_{\Lambda_\pm} \hookrightarrow \mathcal{L}_{\Lambda_\pm}$$

is homotopic to the constant map. Recall that by definition, both $\Sigma_{\Lambda_\pm}$ and $\Lambda_{\Lambda_\pm}$ are the subsets of the paths connecting Λ_- and Λ_+ in $\Sigma_k(\mathbb{R}^n)$ and $\Lambda(n)$ respectively. Hence, the above assertion will be an immediate consequence of the following theorem.

Theorem 5.15. *For each (open) stratum* $\Sigma_k(\mathbb{R}^n)$*, the inclusion map*

$$j : \Sigma_k(\mathbb{R}^n) \hookrightarrow \Lambda(n)$$

is homotopic to the constant map. In other words, each stratum $\Sigma_k(\mathbb{R}^n)$ *is contractible to a point in* $\Lambda(n)$.

Remark 5.16. We would like to emphasize that *the space* $\Sigma_k(\mathbb{R}^n)$ *itself is not contractible.* In fact, it is a fiber bundle over $\mathrm{Gr}_k(\mathbb{R}^n)$ with fiber $\Lambda_0(n-k) \cong \mathrm{Sym}\,(\mathbb{R}^{n-k})$. Therefore $\Sigma_k(\mathbb{R}^n)$ is a deformation retract to $\mathrm{Gr}_k(\mathbb{R}^n)$. Only when $k = 0$ or $k = n$, the $\Lambda_k(\mathbb{R}^n)$ is contractible. Although the latter is a well-known fact, Theorem 5.15 does not seem to be known previously in the literature, as far as we know.

To prove Theorem 5.15, we need some preliminary known facts on the Lagrangian Grassmannian $\Lambda(n)$ (e.g., see [3]).

Definition 5.17. The *train* of a given point of $\Lambda(n)$ is the set of all Lagrangian subspaces which are not transverse to the given one. The given point is called the *vertex* of the train.

For example, the train of $\mathbb{R}^n \subset \mathbb{C}^n$ is just the standard Maslov cycle $\overline{\Lambda_1(n)} = \overline{\Sigma_1(\mathbb{R}^n)}$.

Definition 5.18. We call *positive vectors* on $\Lambda(n)$ the velocity vectors of the motions of Lagrangian planes under the action of the systems with positive-definite quadratic Hamiltonians.

It was proven in [3] that positive vectors do not belong to the tangent cone of any train, and the set of positive vectors at $V \in \Lambda(n)$ forms a cone in $T_V\Lambda(n)$. We denote this cone by $C_V^+\Lambda(n) \subset T_V\Lambda(n)$, and by $C^+(\Lambda(n))$ the union of these cones over $\Lambda(n)$, which becomes a fiber bundle when restricted to each stratum $\Sigma_k(\mathbb{R}^n)$. We denote this cone bundle by $C^+(\Sigma_k(\mathbb{R}^n)) \to \Sigma_k(\mathbb{R}^n)$. The following lemma enables us to prove Theorem 5.15.

Lemma 5.19. *The bundle $C^+(\Sigma_k(\mathbb{R}^n)) \to \Lambda(n)$ has a canonical section defined by*

$$s(V) = \frac{d}{d\theta}\Big|_{\theta=0} e^{i\theta} \cdot V,$$

where we consider $\theta \mapsto e^{i\theta}V$ as a curve in $\Lambda(n)$.

Proof . This is obvious because the curve is generated by the Hamiltonian $H = \frac{1}{2}(\sum_{i=1}^n |z_i|^2)$. q.e.d.

Recall that the tangent space of $\Lambda(n)$ at any point V can be canonically identified with the quadratic forms on V. We denote this quadratic form $Q(\widehat{V} : V)$ for each $\widehat{V} \in T_V\Lambda(n)$. Then the following lemma can be easily proven from the definition of $\Sigma_k(\mathbb{R}^n)$ and the positive vectors (see [3] for a proof.)

Lemma 5.20. *At $V \in \Sigma_k(\mathbb{R}^n)$, the form $Q(s(V) : V)$ is positive definite on $V \cap \mathbb{R}^n$.*

Now we are ready to prove Theorem 5.14.

Proof of Theorem 5.15. We first choose a compact set $K \subset \Sigma_k(\mathbb{R}^n)$ that is a deformation retract of $\Sigma_k(\mathbb{R}^n)$. Such a compact subset exists because $\Sigma_k(\mathbb{R}^n)$ has a fibering over $\mathrm{Gr}_k(R^n)$ with the fiber $\Lambda_0(n-k) \cong \mathrm{Sym}(\mathbb{R}^{n-k})$. Note that $\mathrm{Sym}(\mathbb{R}^{n-k})$ is contractible. Denote this deformation by $F_t : \Sigma_k(\mathbb{R}^n) \to \Sigma_k(\mathbb{R}^n)$. Now considering $s : \Sigma_k(\mathbb{R}^n) \to C^+(\Sigma_k(\mathbb{R}^n)) \subset T\Lambda(n)|_{\Sigma_k(\mathbb{R}^n)}$ as a vector field along $\Sigma_k(\mathbb{R}^n)$ in $\Lambda(n)$, we can define a smooth deformation of K in $\Lambda(n)$ $G : K \times [0,1] \to \Lambda(n)$ so that for all $V \in K$

$$G(V,0) = j(V) \quad \text{and} \quad \frac{\partial G}{\partial t}\Big|_{t=0} = s(V).$$

By Lemma 5.20 and the compactness of K, there exists some $\epsilon > 0$ such that

$$G(V,t) \in \Sigma_0(\mathbb{R}^n)$$

for all $V \in K$ and $0 < s \leq \epsilon$. Recall $\Sigma_0(\mathbb{R}^n)$ is contractible, and denote a homotopy to a point V_0 by $H : \Sigma_0(\mathbb{R}^n) \times [0,1] \to \Lambda(n)$ with $H(\cdot, 0) = id$ and $H(\cdot, 1) = V_0 \in \Lambda(n)$ where V_0 is a fixed element in $\Lambda(n)$. Now we compose these homotopies to obtain a homotopy $L : \Sigma_k(\mathbb{R}^n) \times [0, 2+\epsilon] \to \Lambda(n)$ by

$$L(V,t) = \begin{cases} j \circ F(V,t) & \text{for} \quad 0 \leq t \leq 1, \\ G(F(V,1), t-1) & \text{for} \quad 1 \leq t \leq 1+\epsilon, \\ H(G_\epsilon \circ F_1(V), t-(1+\epsilon)) & \text{for} \quad 1+\epsilon \leq t \leq 2+\epsilon, \end{cases}$$

which is a homotopy from $j : \Sigma_k(\mathbb{R}^n) \hookrightarrow \Lambda(n)$ to the constant map V_0. This finishes the proof. q.e.d.

6. General construction of symplectic invariants

¿From now on to the end of this paper, we will fix the canonical coherent orientation $\sigma \in \mathrm{Or}([S] : M)$ so that $HF_*^\sigma(H, S, J : M)$ is canonically isomorphic to the singular homology $H_*(S, \mathbb{Z})$. With this, we will also suppress σ from the notation $HF_*^\sigma(H, S, J : M)$.

In this section, we give the general construction of certain symplectic invariants based on the machinery developed in Part I. With regard to the critical point theory, this general construction should be regarded as a *direct approach* for using the Floer theory to detect the linking properties of the mini-maxing sets; we will use this construction below to select the corresponding critical values. We will be interested in the most primitive form of the invariants in this paper and postpone the construction and applications of more refined invariants in the future works.

We first note that the equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\ u(\tau, 0) \in o_M, \\ u(\tau, 1) \in N^*S \end{cases} \tag{6.1}$$

is the negative gradient flow of $\mathcal{A}_H|_{\Omega(S)}$ with respect to the metric $\langle\langle\ ,\ \rangle\rangle_J$ on $\Omega(S)$ and so preserves the downward filtration given by the values of the action functional $\mathcal{A}_H$. In other words, the map

$$\tau \mapsto \mathcal{A}_H(u(\tau))$$

is monotonically decreasing for any solution u of (6.1). This fact is analytically encoded in the identity

$$\frac{d}{d\tau}\mathcal{A}_H(u(\tau)) = -\int |\frac{\partial u}{\partial t} - X_H(u)|_J^2 dt. \tag{6.2}$$

Let $S \subset M$ be a given compact submanifold and let $(H, J) \in \mathcal{N}_{\mathrm{reg}}(S)$ be defined as in (4.6). For $a \in \mathbb{R}$, we define:

$$\begin{aligned} CF^a(H, S : M) &= \{z \in \mathrm{Crit}\mathcal{A}_H|_{\Omega(S)} \mid \mathcal{A}_H(z) < a\} \\ &= \{z : [0,1] \to T^*M \mid z(0) \in o_M, z(1) \in N^*S, \dot{z} = X_H(z) \\ &\qquad\qquad \text{and} \quad \mathcal{A}_H(z) < a \}, \end{aligned}$$

CF_*^a = the $\mathbb{Z}$-free module generated by them, and

$$CF_*^{[a,b)} = CF_*^b / CF_*^a.$$

Then the boundary map, defined in (5.7),

$$\partial_{(H,J)} : CF_*(H, S : M) \to CF_*(H, S : M)$$

induces the (relative) boundary map

$$\partial_{(H,J)} = \partial_{(H,J)}^{[a,b)} : CF_*^{[a,b)}(H, S : M) \to CF_*^{[a,b)}(H, S : M)$$

for any $b > a$, which will obviously satisfy

$$\partial_{(H,J)}^{[a,b)} \circ \partial_{(H,J)}^{[a,b)} = 0.$$

Hence, we can define the relative homology groups by

$$HF_*^{[a,b)}(H, S, J : M) := \text{ Ker } \partial_{(H,J)}^{[a,b)} / \text{ Im } \partial_{(H,J)}^{[a,b)}. \tag{6.3}$$

¿From the definition, there is a natural homomorphism

$$j_* : HF_*^{[a,b)} \to HF_*^{[c,d)}$$

when $a \le c$ and $b \le d$. In particular, there exists a natural homomorphism

$$j_*^\lambda : HF_*^{(-\infty,\lambda)} \to HF_* = HF_*^{(-\infty,\infty)}. \tag{6.4}$$

Definition 6.1. Let $S \subset M$ be a compact submanifold, and let $(H, J) \in \mathcal{N}_{\text{reg}}(S)$. We define the real number $\rho(H, S, J)$ by

$$\rho(H, S, J) := \inf_{\lambda}\{\lambda \in \mathbb{R} \mid j_*^\lambda : HF_*^{(-\infty,\lambda)}(H, S, J : M) \\ \to HF_*(H, S, J : M), \text{ is surjective}\}$$

Lemma 6.2. *For (H, S, J) as in Definition 6.1, $\rho(H, S, J)$ is a finite number which becomes a critical value $\mathcal{A}_H|_{\Omega(S)}$.*

Proof. Since $N^*S \pitchfork \phi_H^1(o_M)$, there are only finite many solutions of

$$\begin{cases} \dot{z} = X_H(z), \\ z(0) \in o_M, z(1) \in N^*S, \end{cases}$$

i.e., there are finitely many critical points and so finitely many critical values of $\mathcal{A}_H|_{\Omega(S)}$. We recall from Theorem 5.5 that

$$HF_*(H, S, J : M) \cong H_*(S, \mathbb{Z}) \neq \{0\}. \tag{6.5}$$

Furthermore since there are only finitely many critical values and in particular the set of critical values is bounded, we have

$$HF_*^{(-\infty,\lambda)}(H, S, J : M) = \{0\} \tag{6.6}$$

for sufficiently negative λ. Combining (6.5) and (6.6), we immediately derive

$$\rho(H, S, J) > -\infty$$

from the definition. Again from the boundedness of the critical values, the inclusion homomorphism

$$j_*^\lambda : HF_*^{(-\infty,\lambda)}(H, S, J : M) \to HF_*(H, S, J : M)$$

becomes an isomorphism if $\lambda \geq K$ for sufficiently large $K \in \mathbb{R}$. This proves

$$\rho(H, S, J) < \infty.$$

Finally the fact that the finite value $\rho(H, S, J)$ is a critical value easily follows from the fact that for $\lambda_2 > \lambda_1$, the natural map

$$j_* : HF_*^{(-\infty,\lambda_1)}(H, S, J : M) \to HF_*^{(-\infty,\lambda_2)}(H, S, J : M)$$

is an isomorphism as long as there are no critical values of $\mathcal{A}_H|_{\Omega(S)}$ with

$$\lambda_1 \leq \mathcal{A}_H(z) \leq \lambda_2.$$

This finishes the proof of Lemma. q.e.d.

Next, we study the J-dependence of $\rho(H, S, J)$ for fixed S and $H \in \mathcal{H}_S$ when J varies among $\mathcal{J}_{(S,H)}$.

Lemma 6.3. *Let $J^\alpha, J^\beta \in \mathcal{J}_{(S,H)}$. Then we have*

$$\rho(H, S, J^\alpha) = \rho(H, S, J^\beta).$$

Proof . Using the fact that $\mathcal{J}^c$ is contractible and in particular connected, we can choose a path $\overline{J} = \{J^s\}_{0 \leq s \leq 1}$ in $\mathcal{J}^c$ connecting J^α and J^β so that the solution set of

$$\begin{cases} \frac{\partial u}{\partial \tau} + J^{\rho_K(\tau)}\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0, \\ u(\tau, 0) \in o_M, u(\tau, 1) \in N^*S \end{cases} \tag{6.7}$$

satisfies the regular property required before, provided $K > 0$ is sufficiently large. Recall that the canonical homomorphism

$$h_{\alpha\beta} : CF_*(H, S, J^\alpha) \to CF_*(H, S, J^\beta)$$

is defined by

$$h_{\alpha\beta}(z^\alpha) = \sum n^{\alpha\beta}(z^\alpha, z^\beta) z^\beta,$$

where $n^{\alpha\beta}(z^\alpha, z^\beta) = \#(\mathcal{M}_K(z^\alpha, z^\beta))$ that induces an isomorphism

$$HF_*(H, S, J^\alpha : M) \to HF_*(H, S, J^\beta : M).$$

To see how $\rho(H, S, J)$ vary under the change of J, we need to estimate the difference

$$\mathcal{A}_H(z^\beta) - \mathcal{A}_H(z^\alpha)$$

whenever $n^{\alpha\beta}(z^\alpha, z^\beta) \neq 0$ and so in particular when there exists a solution u of (6.7) with

$$\lim_{\tau \to -\infty} u(\tau) = z^\alpha, \lim_{\tau \to \infty} u(\tau) = z^\beta.$$

For such u, we then write

$$\mathcal{A}_H(z^\beta) - \mathcal{A}_H(z^\alpha) = \int_{-\infty}^{\infty} \frac{d}{d\tau} \mathcal{A}_H(u(\tau)) d\tau. \tag{6.8}$$

However, we have

$$\begin{aligned}\frac{d}{d\tau}\mathcal{A}_H(u(\tau)) &= d\mathcal{A}_H(u(\tau))\cdot\frac{du}{d\tau}\\ &= \int_0^1 \left(\omega(\frac{\partial u}{\partial t},\frac{\partial u}{\partial \tau}) - dH_t(u)\frac{\partial u}{\partial \tau}\right) \quad \text{from (2.17)}\\ &= \int_0^1 \langle J^{\rho(\tau)}(\frac{\partial u}{\partial t} - X_H(u)), \frac{\partial u}{\partial \tau}\rangle_{J_t^{\rho(\tau)}}\\ &= -\int_0^1 |\frac{\partial u}{\partial t} - X_H(u)|^2_{J_t^{\rho(\tau)}} \quad \text{from the equation (6.7)}\\ &\leq 0.\end{aligned}$$

Note that the boundary term from (2.17) drops out due to the *fixed* boundary condition required in (6.7). Hence, we have proved that whenever there exists a solution u as above, we have

$$\mathcal{A}_H(z^\beta) \leq \mathcal{A}_H(z^\alpha).$$

This shows that the map $h_{\alpha\beta}: CF_*(H,S,J^\alpha) \to CF_*(H,S,J^\beta)$ restricts to a map

$$h_{\alpha\beta}: CF_*^{(-\infty,\lambda)}(H,S,J^\alpha) \to CF_*^{(-\infty,\lambda)}(H,S,J^\beta)$$

for any $\lambda \in \mathbb{R}$ and so induces a homomorphism

$$h_{\alpha\beta}: HF_*^{(-\infty,\lambda)}(H,S,J^\alpha : M) \to HF_*^{(-\infty,\lambda)}(H,S,J^\beta : M).$$

Now consider the commutative diagram

$$\begin{array}{ccc} HF_*^{(-\infty,\lambda)}(H,S,J^\alpha : M) & \overset{(j_*^\lambda)_\alpha}{\to} & HF_*(H,S,J^\alpha : M)\\ \downarrow h_{\alpha\beta} & & \downarrow h_{\alpha\beta}\\ HF_*^{(-\infty,\lambda)}(H,S,J^\beta : M) & \underset{(j_*^\lambda)_\beta}{\to} & HF_*(H,S,J^\beta : M).\end{array}$$

Since $h_{\alpha\beta}$ on the right-hand side is an isomorphism, if $(j_*^\lambda)_\alpha$ is surjective, so is $(j_*^\lambda)_\beta$. Therefore, from the definition we have proved

$$\rho(H,S,J^\alpha) \geq \rho(H,S,J^\beta).$$

Changing the roles of α and β yields

$$\rho(H,S,J^\beta) \geq \rho(H,S,J^\alpha),$$

which finishes the proof of $\rho(H,S,J^\alpha) = \rho(H,S,J^\beta)$ q.e.d.

Lemma 6.3 now allows us to define the invariant $\rho(H,S,J)$ for *any* $J \in \mathcal{J}^c$ by simply extending the definition to all $\mathcal{J}^c$ by continuity.

Definition 6.4. Let $S \subset M$ be a compact manifold and $H \in \mathcal{H}_S$. We define

$$\rho(H,S) := \rho(H,S,J)$$

for a $J \in \mathcal{J}(S,H)$ (and so for any $J \in \mathcal{J}$).

Next, we study the dependence of $\rho(H,S)$ on (H,S). Since our primary interest is the study of a given Lagrangian submanifold $\phi_H^1(o_M)$, we will fix $H \in \mathcal{H}$ and vary S first.

Proposition 6.5. *Let $S_0 \subset M$ and let* Iso $(S_0 : M)$ *be the isotopy class of S_0 in M. Then the assignment*

$$S^\alpha \mapsto \rho(S^\alpha, H)$$

on $S^\alpha \in \mathrm{Iso}^H(S_0 : M)$ is continuous on S^α in the C^1-topology of Iso $(S_0 : M)$. *Hence we can extend the definition of $\rho(S,H)$ to all $S \in$* Iso $(S_0 : M)$ *by continuity in C^1-topology of* Iso $(S_0 : M)$.

Proof . The idea of the proof of this proposition is similar to that of Lemma 6.3. Similar arguments will appear again and again in this paper. Let S^α and $S^\beta \in \mathrm{Iso}^H(S_0 : M)$ and let S^s be a generic isotopy between them. It will be enough to consider the case where S^α and S^β are sufficiently C^1-close so that the map $\phi : S^\alpha \to S^\beta$ defined by the nearest point becomes a diffeomorphism. We now consider the equation for $J \in \mathcal{H}_{(H,S^\alpha)} \cap \mathcal{H}_{(H,S^\beta)}$

$$\text{(6.9)} \qquad \begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\ u(\tau,0) \in o_M, u(\tau,1) \in N^*S^{\rho_K(\tau)}, \\ \lim_{\tau\to-\infty} u(\tau) = z^\alpha, \quad \lim_{\tau\to\infty} u(\tau) = z^\beta \end{cases}$$

for each $z^\alpha \in \mathcal{M}_J(H,S^\alpha)$ and $z^\beta \in \mathcal{M}_J(H,S^\beta)$. As before, we estimate

$$\mathcal{A}_H(z^\beta) - \mathcal{A}_H(z^\alpha)$$

for the pair (z^α, z^β) with $n^{\alpha\beta}(z^\alpha, z^\beta) \neq 0$ (and so (6.9) has a solution). This time from (2.17) we have

$$\begin{aligned}\frac{d}{d\tau}\mathcal{A}_H(u(\tau)) &= d\mathcal{A}_H(u(\tau)) \cdot \frac{du}{d\tau} \\ &= -\int_0^1 |\frac{\partial u}{\partial t} - X_H(u)|^2_{J_t} dt + \langle T\pi \frac{\partial u}{\partial \tau}(\tau, 1), u(\tau, 1)\rangle \\ &\leq \langle \frac{\partial}{\partial \tau}(\pi \circ u)(\tau, 1), u(\tau, 1)\rangle.\end{aligned}$$

Recall that $u(\tau, 1) \in N^*S^{\rho_K(\tau)}$ and so $\pi \circ u(\tau, 1) \in S^{\rho(\tau)}$. Therefore, the component $(\frac{\partial}{\partial \tau}(\pi \circ u)(\tau, 1))^\perp$ to $S^{\rho(\tau)}$ has the inequality

$$\Big|(\frac{\partial}{\partial \tau}(\pi \circ u)(\tau, 1))^\perp\Big| \leq \rho'(\tau)\Big|\frac{\partial S^s}{\partial s}\Big|. \tag{6.10}$$

Furthermore, by the C^0-estimate (3.8), we also have

$$|u(\tau, 1)|_g \leq C, \tag{6.11}$$

where $|u(\tau, 1)|_g$ is the norm as an element in $T^*_{\pi(u(\tau,1))}M$. Combining (6.10) and (6.11), and integrating $\int_{-\infty}^{\infty} \frac{d}{d\tau}\mathcal{A}_H(u(\tau))d\tau$, we get

$$\begin{aligned}\mathcal{A}_H(z^\beta) - \mathcal{A}_H(z^\alpha) &\leq C\int_{-\infty}^{\infty} \rho_K'(\tau)\Big|\frac{\partial S^s}{\partial s}\Big| d\tau \\ &\leq C \max_{s\in[0,1]}\Big|\frac{\partial S^s}{\partial s}\Big| \int_{-\infty}^{\infty} \rho_K'(\tau) d\tau \\ &= C \max_{s\in[0,1]}\Big|\frac{\partial S^s}{\partial s}\Big|.\end{aligned}$$

Obviously, one can choose the path $\{S^s\}_{0\leq s\leq 1}$ so that

$$\max_{s\in[0,1]}\Big|\frac{\partial S^s}{\partial s}\Big| \sim d_{C^1}(S^\alpha, S^\beta),$$

and hence,

$$\mathcal{A}_H(z^\beta) - \mathcal{A}_H(z^\alpha) \leq C d_{C^1}(S^\alpha, S^\beta) := \epsilon^{\alpha\beta},$$

when S^α and S^β are C^1-close. As before, the natural homomorphism $h_{\alpha\beta} : HF_*(H, S^\alpha, J : M) \to HF_*(H, S^\beta, J : M)$ induces the commutative diagram:

$$\begin{array}{ccc} HF_*^{(-\infty,\lambda)}(H,S^\alpha,J:M) & \stackrel{j_*^\lambda}{\longrightarrow} & HF_*(H,S^\alpha,J:M) \\ \downarrow h_{\alpha\beta} & & \downarrow h_{\alpha\beta} \\ HF_*^{(-\infty,\lambda+\epsilon^{\alpha\beta})}(H,S^\beta,J:M) & \underset{j_*^{\lambda+\epsilon^{\alpha\beta}}}{\longrightarrow} & HF_*(H,S^\beta,J:M) \end{array}$$

Again since $h_{\alpha\beta}$ on the right-hand side is an isomorphism, we conclude

$$\rho(S_\beta,H) \le \rho(S_\alpha,H) + Cd_{C^1}(S^\alpha,S^\beta).$$

By changing the roles of α and β, we prove the other side of the inequality and so

$$|\rho(S^\beta,H) - \rho(S^\alpha,H)| \le Cd_{C^1}(S^\alpha,S^\beta),$$

which in particular proves the continuity of $\rho(\cdot,H)$. q.e.d.

Remark 6.6. We would like to emphasize that in the proof of Proposition 7.5, we have used the a priori C^0-estimate (3.8) in an essential way.

In the next section, we will study the most important property of $\rho(H,S)$, the dependence of $\rho(H,S)$ on H.

7. Basic properties of the invariants $\rho(H,S)$

In this section, we fix $S \subset M$ and start with considering H's in $\mathcal{H}_S$. We first prove the following easy lemma.

Lemma 7.1. *When $H \in \mathcal{H}_S$ and $\|H\|_{C^1} \to 0$, then $\rho(H,S) \to 0$.*

Proof . Let z be any solution of

$$\begin{cases} \dot z = X_H(z), \\ z(0) \in o_M, z(1) \in N^*S. \end{cases}$$

Then

$$\begin{aligned} \mathcal{A}_H(z) &= \int z^*\theta - \int_0^1 H(z(t),t)dt \\ &= \int_0^1 \langle T\pi(\dot z), z\rangle - \int_0^1 H(z(t),t)dt \end{aligned}$$

and hence

$$|\mathcal{A}_H(z)| \le \int_0^1 |T\pi(\dot z)||z| + \int_0^1 \max_x |H| dt.$$

By the equation $\dot z = X_H(z)$ it is immediately seen that

$$|\mathcal{A}_H(z)| \le C \int_0^1 \max_x (|dH|^2 + |H|) dt \tag{7.1}$$

for all solutions z. Using the fact that $\rho(H,S)$ is a critical value and so $\rho(H,S) = \mathcal{A}_H(z)$ for some solution z of the equation, the lemma follows from (7.1). q.e.d.

The following theorem summarizes the basic properties of $\rho(H,S)$.

Theorem 7.2. *Let $S \subset M$ be a compact manifold and assume that $H, H^\alpha, H^\beta \in \mathcal{H}_S$. Then,* (1) *We have*

$$\begin{aligned} \int_0^1 -\max_x\, (H^\beta - H^\alpha) dt \le & \rho(H^\beta, S) - \rho(H^\alpha, S) \\ & \le \int_0^1 -\min_x\, (H^\beta - H^\alpha) dt, \end{aligned} \tag{7.2}$$

which particularly together with Lemma 7.1 leads to

$$\int_0^1 -\max_x H dt \le \rho(H,S) \le \int_0^1 -\min_x H dt. \tag{7.3}$$

(2) *From (1) we obtain*

$$|\rho(H^\beta, S) - \rho(H^\alpha, S)| \le \|H_\beta - H_\alpha\|_{C^0},$$

which in particular implies that for fixed S, one can extend the assignment $H \mapsto \rho(H,S)$ to all $\mathcal{H}$ as a continuous function in the C^0-topology of $\mathcal{H}$. We will continue to denote the extension by $\rho(H,S)$.

Remark 7.3. By combining Proposition 7.5 and Theorem 7.2 (2) we can now extend the definition of $\rho(H,S)$ to the set

$$\mathcal{H}_{C^0} \times \ \mathrm{Iso}_{C^1}(S_0 : M),$$

where

$\mathcal{H}_{C^0}$ = the set of asymptotically constant C^0-functions on $T^*M \times [0,1]$,

$\mathrm{Iso}_{C^1}(S_0 : M)$ = the set of C^1-embeddings which are isotopic to S_0.

In fact, one can see that we can even extend the definition to

$$\mathcal{H}_{C^0} \times \ \mathrm{Iso}_{\mathrm{Lip}}\ (S_0 : M).$$

It would be interesting to study the geometric meaning of $\rho(H, S)$ for the cases where $H \in C^0$ but not in C^1.

Proof. The proof of (2) immediately follows from (7.2) and so we need only to prove (1). Consider the linear homotopy

$$H^s := (1-s)H^\alpha + sH^\beta.$$

Although this homotopy may not be regular in the sense of Theorem 4.2 (1), we will pretend it is so for the moment and explain the necessary justification in the end. Consider the equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_{H_K^{\rho(\tau)}}(u)) = 0, \\ u(\tau, 0) \in o_M, u(\tau, 1) \in N^*S, \\ \lim_{\tau \to -\infty} u(\tau) = z^\alpha,\ \lim_{\tau \to \infty} u(\tau) = z^\beta. \end{cases} \tag{7.4}$$

For the notational convenience, we just denote $\rho = \rho_K$ below. As before, for the pair (z^α, z^β) with $n_{\alpha\beta}(z^\alpha, z^\beta) \neq 0$ we compute

$$\mathcal{A}_{H^\beta}(z^\beta) - \mathcal{A}_{H^\alpha}(z^\alpha) = \int_{-\infty}^{\infty} \frac{d}{d\tau}\Big(\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))\Big) d\tau,$$

and

$$\frac{d}{d\tau}(\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))) = d\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))(\frac{du}{d\tau}) - \int_0^1 (\tfrac{\partial H^{\rho(\tau)}}{\partial \tau})(u, t) dt.$$

Here as before,

$$d\mathcal{A}_{H^{\rho(\tau)}}(u(\tau))(\frac{du}{d\tau}) = -\int_0^1 |\frac{\partial u}{\partial t} - X_{H^{\rho(\tau)}}(u)|_J^2 dt \leq 0,$$

$$\begin{aligned} \int_0^1 (\tfrac{\partial H^{\rho(\tau)}}{\partial \tau})(u, t) dt &= -\int_0^1 \rho'(\tau)(H^\beta - H^\alpha)(u, t) dt \\ &\leq -\rho'(\tau) \int_0^1 \min_x (H^\beta - H^\alpha) dt, \end{aligned} \tag{7.5}$$

and hence,

$$\begin{aligned}\mathcal{A}_{H^\beta}(z^\beta) - \mathcal{A}_{H^\alpha}(z^\alpha) &\leq \int_{-\infty}^{\infty} -\rho'(\tau) \int_0^1 \min_x (H^\beta - H^\alpha) dt d\tau \\ &= \int_0^1 - \min_x (H^\beta - H^\alpha) dt \int_{\infty}^{-\infty} \rho'(\tau) d\tau \\ &= \int_0^1 - \min_x (H^\beta - H^\alpha) dt.\end{aligned}$$

By the similar arguments as before this estimate implies

$$\rho(H^\beta, S) \leq \rho(H^\alpha, S) + \int_0^1 - \min_x (H^\beta - H^\alpha) dt,$$

i.e.,

$$\rho(H^\beta, S) - \rho(H^\alpha, S) \leq \int_0^1 - \min_x (H^\beta - H^\alpha) dt \tag{7.6}$$

by considering the homomorphism

$$h_{\alpha\beta} : HF_*^{(-\infty,\lambda)}(H^\alpha, S, J : M) \to HF_*^{(-\infty,\lambda+\epsilon^{\alpha\beta})}(H^\beta, S, J : M),$$

where $\epsilon^{\alpha\beta} = -\int_0^1 \min_x (H^\beta - H^\alpha) dt$. Changing the roles of α and β also leads to

$$\rho(H^\alpha, S) \leq \rho(H^\beta, S) + \int_0^1 - \min_x (H^\alpha - H^\beta) dt,$$

i.e.,

$$\begin{aligned}\rho(H^\beta, S) - \rho(H^\alpha, S) &\geq \int_0^1 \min_x (H^\alpha - H^\beta) dt \\ &= \int_0^1 - \max_x (H^\beta - H^\alpha) dt,\end{aligned} \tag{7.7}$$

where we have used the identity

$$- \max_x f(x) = \min_x (-f(x)).$$

Combining (7.6) and (7.7), we will have finished the proof if we can justify the use of the linear homotopy which might not be regular. To do this, we proceed as follows. For each given $\epsilon > 0$, we approximate the above linear homotopy by C^1-close regular homotopies H so that for all $t \in [0,1]$

$$\max_{x,s} \left| \frac{\partial H}{\partial s}(x,t,s) - (H^\beta - H^\alpha)(x,t) \right| \leq \epsilon. \tag{7.8}$$

Then for this homotopy, (7.5) will be replaced by

$$\begin{aligned} -\int_0 (\frac{\partial H^{\rho(\tau)}}{\partial \tau})(u,t)dt &= -\int_0^1 \rho'(\tau)\frac{\partial H}{\partial s}(u,t,\rho(\tau))dt \\ &\leq \rho'(\tau)\int_0^1 -\min_{x,s}\frac{\partial H}{\partial s}(s,t,s)dt \\ &\leq \rho'(\tau)(\int_0^1 -\min_{x,s}(H^\beta - H^\alpha) + \epsilon)dt, \end{aligned}$$

which implies

$$\mathcal{A}_{H^\beta}(z^\beta) - \mathcal{A}_{H^\alpha}(z^\alpha) \leq \left(\int_0^1 -\min_x(H^\beta - H^\alpha) + \epsilon\right) dt.$$

By letting $\epsilon \to 0$, we are done for (7.2). To prove (7.3), we set $H^\beta = H$ and $H^\alpha \to 0$ in C^1-topology and apply Lemma 7.6 and (7.2). This finishes the proof. q.e.d.

8. Symplectic invariants of Lagrangian submanifolds

In the previous section, we have defined $\rho(H,S)$ for each pair (H,S). It turns out that $\rho(H,S)$ depends only on the Lagrangian submanifold $L = \phi_H(o_M)$ up to a *universal normalization* independent of $S \subset M$.

To explain the normalization, we recall that the wave front set of exact Lagrangian submanifold L (e.g., $L = \phi_H(o_M)$) is uniquely defined up to the vertical translation on $M \times \mathbb{R}$. We now recall Proposition 2.6: When $H \mapsto L$ is given, we select the wave front of L as

$$W_H = \{(q,r) \mid q = \pi(p), \qquad r = \mathcal{A}_H(z_H^p),\, p \in L\}.$$

Then Proposition 2.6 (1) implies (if we assume L is connected)

$$W_K = W_H + c\frac{\partial}{\partial r}$$

for some $c \in \mathbb{R}$, provided $H, K \mapsto L$, i.e., $\phi_H(o_M) = \phi_K(o_M) = L$. And Proposition 2.6 (2) implies that

$$W_{H+c_0} = W_H + c_0\frac{\partial}{\partial r}$$

for any constant $c_0 \in \mathbb{R}$. The following theorem shows that $\rho(H,S)$ is the invariant of $L = \phi_H(o_M)$ up to the normalization.

Theorem 8.1. *Suppose that $H, K \mapsto L$ and that $W_H = W_K$. Then $\rho(K,S) = \rho(H,S)$ for all $S \subset M$.*

Proof . We will use a transformation between the *geometric* and the *dynamical* versions of the Floer theory in an essential way. To prove the theorem, we have to show that for H, K as in the theorem,

$$\rho(H,S,J) = \rho(K,S,J) \tag{8.1}$$

for any $S \in \mathrm{Iso}^H \cap \mathrm{Iso}^K$ and $J \in \mathcal{J}_{H,S} \cap \mathcal{J}_{K,S}$, and then the theorem will follow by the continuity property of ρ for all S and J.

We first explain some simple but crucial transformation between two versions of the Floer theory, one the *geometric version* used by Floer [17] and the present author [38] previously, and the other the *dynamical version* that is being used in the present paper. In the present paper, we have defined $HF_*(H,S,J:M)$ by considering the solutions of Hamilton's equation

$$\begin{cases} \dot z = X_H(z), \\ z(0) \in o_M, z(1) \in N^*S, \end{cases} \tag{8.2}$$

and the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\ u(\tau,0) \in o_M, \\ u(\tau,1) \in N^*S \end{cases} \tag{8.3}$$

for the path $u : \mathbb{R} \to \Omega(S)$, i.e., a map

$$u : \mathbb{R} \times [0,1] \to T^*M.$$

On the other hand, one can define an equivalent object for a suitably chosen $\widetilde J$ which we denote by $HF_*(\phi_H(o_M), N^*S, \widetilde J)$ considering the intersections $\phi_H(o_M) \cap N^*S$ and the Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + \widetilde J \frac{\partial u}{\partial t} = 0, \\ u(\tau,0) \in \phi_H(o_M), \\ u(\tau,1) \in N^*S \end{cases} \tag{8.4}$$

for the path $u : \mathbb{R} \to \Omega(L, N^*S)$ where $L = \phi_H(o_M)$ and

$$\Omega(L, N^*S) = \{\gamma : [0,1] \to T^*M \mid \gamma(0) \in L,\, \gamma(1) \in N^*S\}.$$

The important feature of both equations is that they are the *gradient flow* of $\mathcal{A}_H|_{\Omega(S)}$ and the Floer's functional $\underline{a}^S$ defined as in (2.27) respectively. An advantage of the geometric version is that it depends only on the Lagrangian submanifold $L = \phi_H(o_M)$ not on H, but as mentioned in Proposition 2.8, one can assume that under a suitable normalization of $\underline{a}^S$ as in Proposition 2.8

$$\mathcal{A}_H(z_H^p) = \underline{a}^S(p) \tag{8.5}$$

for all $p \in \phi_H(o_M) \cap N^*S$. We denote by $\underline{a}_H^S$ this normalized $\underline{a}^S$. The following is an easy but crucial lemma that we shall use.

Lemma 8.2. (1) *The map $\Phi_H : \Omega(S) \to \Omega(L, N^*S)$ defined by*

$$\gamma \mapsto \phi_H^1(\phi_H^t)^{-1}\gamma$$

gives rise to the one-one correspondence between the set $\phi_H(o_M) \cap N^*S \subset \Omega(L, N^*S)$ as constant paths and the set of solutions of (7.12).

(2) *The map $u \mapsto \Phi_H(u)$ also defines a one-one correspondence from the set of solutions of (8.3) and that of*

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial \tau} + J^H \frac{\partial \widetilde{u}}{\partial t} = 0, \\ \widetilde{u}(\tau, 0) \in \phi_H(o_M), \\ \widetilde{u}(\tau, 1) \in N^*S, \end{cases} \tag{8.6}$$

where $J^H = \{J_t^H\}$, $J_t^H := (\phi_H^1(\phi_H^t)^{-1})_* J_t$. Furthermore, (8.3) *is regular if and only if* (8.6) *is regular.*

We will omit the proof by referring to [41, Appendix] for the proof of a similar statement. One immediate corollary of this lemma is

Corollary 8.3. *For the regular H, S and J, the isomorphism*

$$(\Phi_H)_* : HF_*(H, S, J : M) \to HF_*(L, N^*S, J^H)$$

preserves filtrations, i.e., restricts to the isomorphism

$$(\Phi_H)_* : HF_*^{(-\infty,\lambda)}(H, S, J : M) \to HF_*^{(-\infty,\lambda)}(L, N^*S, J^H),$$

where the filtration on the left-hand side is given by $\mathcal{A}_H$ and that on the right is given by $\underline{a}_H^S$.

In the same way as we defined $\rho(H,S,J)$, we can define, for the regular $\widetilde{J}$ (in the sense that all the solutions of (8.4) are regular),

$$\widetilde{\rho}(L,S,\widetilde{J}) = \inf_{\lambda}\{\lambda \mid \widetilde{j}^{\lambda}_{*} : HF_*^{(-\infty,\lambda)}(L,N^*S,\widetilde{J}) \to HF_*(L,N^*S,\widetilde{J}) \text{ is surjective }\}.$$

Furthermore one can also prove in the same way as in the proof of Lemma 7.3 that for a *fixed* normalization of $\underline{a}^S$

$$\widetilde{\rho}(L,S,\widetilde{J}^{\alpha}) = \widetilde{\rho}(L,S,\widetilde{J}^{\beta}) \tag{8.7}$$

for regular $\widetilde{J}^{\alpha}$ and $\widetilde{J}^{\beta}$. Now, we go back to the proof of (8.1). Corollary 8.3 implies that under the normalization of $\underline{a}^S$ in (8.5), we have

$$\rho(H,S,J) = \widetilde{\rho}(L,S,J^H). \tag{8.8}$$

Similarly one can normalize $\underline{a}^S$ so that

$$\rho(K,S,J) = \widetilde{\rho}(L,S,J^K), \tag{8.9}$$

and denote by $\underline{a}^S_K$ the corresponding function. Now, we have only to show that the two normalizations used for H and K agree and then (8.7) applied to J^H and J^K will finish the proof. To prove $\underline{a}^S_H \equiv \underline{a}^S_K$, it is enough to prove

$$\underline{a}^S_H(p) = \underline{a}^S_K(p) \tag{8.10}$$

for some $p \in \phi_H(o_M) \cap N^*S \subset \Omega(L,N^*S)$. However from (8.5) we have

$$\mathcal{A}_H(z^p_H) = \underline{a}^S_H(p),$$
$$\mathcal{A}_K(z^p_K) = \underline{a}^S_K(p)$$

for all $p \in L \cap N^*S$. We just pick any point among them. Now the hypothesis $W_H = W_K$ implies that

$$\mathcal{A}_H(z^p_H) = \mathcal{A}_K(z^p_K).$$

Combining these, we have $\underline{a}^S_H(p) = \underline{a}^S_K(p)$ and so $\underline{a}^S_H \equiv \underline{a}^S_K$. This finishes the proof of (8.1) and so the proof of Theorem 8.1. q.e.d.

Remark 8.4. One might attempt to prove (8.1) by the more familiar continuity argument of finding a path $\{H_s\}$ from H to K and of using Proposition 2.7, the fact that $\mathrm{Spec}(H,S)$ is nowhere dense subset of $\mathbb{R}$. To apply this argument, we have to make the sets $\mathrm{Spec}(H_s, S)$ fixed for all s. However besides the normalization problem which can be solved easily as before, this approach faces a difficult problem of the connectivity question: *Whether the set of Hamiltonians H generating a fixed Lagrangian submanifold L is connected or not.* This is exactly the reason why we bypassed this difficult question using the above transformation between the geometric and the dynamical versions of the Floer theory. One might consider this transformation the analogoue to the *gauge invariance* of the generating function approach.

Now, Theorem 8.1 allows us to define various kinds of homotopy theoretical invariants of the Lagrangian submanifold $L = \phi_H(o_M)$. We call these invariants *capacities of L relative to $S \subset M$.*

Definition 8.5. Let $S_0 \subset M$ be a compact submanifold and $\mathrm{Iso}\,(S_0 : M)$ be the isotopy class of embeddings of S_0. We denote by $[S_0]$ the corresponding isotopy class on M. For a given $L = \phi_H(o_M)$, we define

$$\gamma(L : S_0) := \max_{S \in [S_0]} \rho(H, S) - \min_{S \in [S_0]} \rho(H, S) \tag{8.11}$$

for any $H \mapsto L$, and call it *the capacity of L relative to the class $[S_0]$.*

Note that the right-hand side of (8.11) is independent of the choice of H as long as $H \mapsto L$.

9. Wave front and normalizations

The special case $S = \{pt\}$ is particularly interesting in that it is closely related to the structure of the wave front of the Lagrangian submanifold $L = \phi_H(o_M)$. In this case, for each given H, the assignment

$$q \mapsto \rho(H, \{q\}) \quad \text{on} \quad M$$

defines a continuous function on M, which is a consequence of Proposition 7.5. We denote this function by $f_H : M \to \mathbb{R}$, i.e.,

$$f_H(q) = \rho(H, \{q\})$$

and call it the *basic phase function* of H or of $L = \phi_H(o_M)$. The assignment

$$H \mapsto f_H : \quad \mathcal{H} \to C^0(M)$$

defines a continuous map with respect to the C^0-topology of $\mathcal{H}$ and f_H. Furthermore, it has the property

$$\gamma(L:\{pt\}) = \mathrm{osc} f_H \leq \|H\|, \tag{9.1}$$

where $\|H\|$ is the Hofer's norm. By taking the infimum $\inf_{H\mapsto L}\|H\|$ in (9.1), we have derived the inequality

$$\gamma(L:\{pt\}) \leq d(o_M, L). \tag{9.2}$$

Theorem 9.1. *Let G_{f_H} be the graph of f_H. Then $G_{f_H} \subset M\times\mathbb{R}$ is a subset of the wave front of L, and so f_H is smooth away from a set of codimension at least one, and at smooth points q we have*

$$(q, df_H(q)) \in \phi_H(o_M) = L.$$

Proof . The first statement follows from the definition of $f_H = \rho(H,\{q\})$ and the fact that $f_H(q)$ is a critical value of $\mathcal{A}_H|_{\Omega(\{q\})}$ and so

$$f_H(q) = \mathcal{A}_H(z_H^p) \quad \text{for some} \quad p \in \phi_H(o_M)\cap T_q^*M.$$

The second statement follows from the general property on the wave front set (See [12] for example) because the only possible non-smooth points are those corresponding to the crossings of two different branches of the wave front set of L. q.e.d.

Remark 9.2. In the terminology of [12], the graph G_{f_H} selects a *semi-simple part* (or a graph part) in a canonical way. We suspect that this canonical choice will be useful in the study of structure of the wave front in low dimensions, which should involve finer understanding of the Floer cycles.

Theorem 9.1 gives rise to an easy proof of the nondegeneracy of the Hofer's distance defined in (2.10).

Theorem 9.3. *The distance defined in (2.10) is nondegenerate, i.e., $d(L_1, L_2) = 0$ if and only if $L_1 = L_2$.*

Proof. We first consider the case where $L_1 = o_M$. In this case, we have by definition

$$d(o_M, L_2) = \inf_{H\mapsto L_2}\|H\|.$$

Now, suppose that $d(o_M, L_2) = 0$. Then (9.2) yields

$$\gamma(L_2 : \{pt\}) = 0,$$

i.e., the basic phase function f_H is a constant function on M, and so f_H is smooth everywhere and

$$df_H(q) = 0$$

for all $q \in M$. Therefore, we have proved

$$o_M \subset L_2 = \phi_H(o_M). \tag{9.3}$$

Using the compactness and connectedness of M, it is easy to show that (9.3) indeed implies

$$o_M = L_2,$$

which finishes the proof for the case where $L_1 = o_M$. For the general cases of

$$L_1 = \phi(o_M) \quad \text{and} \quad L_2 = \psi(o_M),$$

we first note that

$$d(L_1, L_2) = d(\eta(L_1), \eta(L_2))$$

for any $\eta \in \mathcal{D}_\omega(T^*M)$. Therefore one can reduce the general case to the special case $L_1 = o_M$. q.e.d.

To illustrate the meaning of the f_H, we give an example for the case where $M = S^1$.

Example 9.4. Let us consider the Lagrangian submanifold $L \subset T^*S$ pictured as in the following figure. Here we denote by z's the intersections of L with the zero section, by x's the caustics and by y the point at which the two shaded regions in the picture have the same area. The corresponding wave front can be easily drawn as

FIGURE 2

FIGURE 3

Note that the points z's correspond to critical points of the action functional, x's to the cusp points of the wave front and y to the point where two different branches of the wave front cross. Using the continuity of the basic phase function f_H where $H \mapsto L$, one can easily see that the graph of f_H is the one bold-lined in Figure 3. We would like to note that the value $\min_{q\in M} F_H(q)$ is *not* a critical value of $\mathcal{A}_H$.

Now, we are ready to provide a *universal normalization* which is continuous with respect to the Hamiltonian H. This will take care of the indeterminacy in defining the invariants of Lagrangian submanifolds $L = \phi_H(o_M)$. We have defined the basic phase function $f_H : M \to K$ by $f_H(q) = \rho(H, \{q\})$, and from the very beginning we have fixed a Riemannian metric g on M and so have the induced measure on M. We define the constant

$$b(H) := \frac{1}{\mathrm{vol}(M)} \int f_H d\mathrm{vol}. \tag{9.4}$$

The following is easy to prove

Lemma 9.5. *For each $H, K \mapsto L$, we have*

$$c(H,K) = b(H) - b(K),$$

where $c(H,K)$ is the constant in (2.23).

Therefore combining Proposition 2.6, Theorem 8.1 and Lemma 8.3, we now define the normalized version of $\rho(H,S)$ by

$$\widetilde{\rho}(H,S) = \rho(H,S) - b(H),$$

which now depends only on the final Lagrangian submanifold $L = \phi_M(o_M)$, and obviously share all the properties similar to Theorem 7.2. We denote the common number by $\rho(L,S) = \widetilde{\rho}(H,S)$.

Definition 9.6. For each $S \subset M$, we define

$$\rho(L,S) = \widetilde{\rho}(H,S)$$

for a $H \mapsto L$ (and so for any H).

Theorem 9.7. *Let $S \subset M$ be a compact submanifold. Then ρ is a continuous function with respect to the Hofer's distance of L and C^1-topology of S.*

Remark 9.8. One could also take

$$b(H) = \min_{x\in M} f_H(x) \quad \text{or} \quad \max_{x\in M} f_H(x)$$

or any distinguished value of f_H, if there is. We would like to note that for Viterbo's invariant, there is no obvious *continuous* universal normalization. In fact, many statements in [56] should be restricted to the case where the Lagrangian submanifolds coincide with the zero section in some neighborhoods of a *fixed* point in terms of the normalization problem. In that case, by assuming that $H \equiv 0$ on the neighborhood, there is the canonical normalization which gives rise to zero as a distinguished value of f_H.

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